Einführung in die Astronomie I
Teil 8

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Übersicht Teil 8

- stellar structure
  - stellar models
  - the main sequence
  - white dwarfs
  - convection
  - Hayashi line

stellar models

- conservation of mass

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

- hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

stellar models

- conservation of energy

$$L_r = \int_0^{M_r} \epsilon \, dM_r = \int_0^r \epsilon 4\pi r^2 \rho \, dr$$

- energy transport (diffusion)

$$\frac{dT}{dr} = -\frac{3}{64\pi} \frac{\chi}{\sigma} \frac{L_r}{r^2 T^3}$$

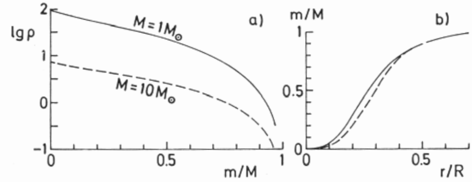
- 4 equations for 4 unknowns!

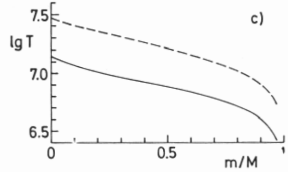
stellar models

- mixed boundary conditions
  - $r = 0$
    - $M_r = 0$
    - $L_r = 0$
  - $r = R$
    - $T \to 0$
    - $P \to 0$
- solution $\to$ numerical!

stellar models

- characteristic variables as functions of m/M:





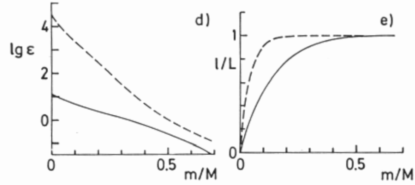


Fig. 22.4. a, b The run of some functions inside zero-age main-sequence models for $M = 1M_\odot$ (*solid lines*) and $M = 10M_\odot$ (*dashed lines*) with the same composition as in Fig. 22.1 ($X_\mathrm{H} = 0.685$, $X_\mathrm{He} = 0.294$); (**a**) density ϱ (in g cm^{-3}), (**b**) radial mass distribution $m(r)$, (**c**) temperature T (in K), (**d**) nuclear energy production (in erg g^{-1} s^{-1}), (**e**) local luminosity l

stellar models

- density rises steeply towards center (9 dex from atmosphere values)
- $10\,M_\odot$ model has 1 dex *lower* central density!
- very strong concentration of mass towards the center of the stars
- $1\,M_\odot$: inner 30% of r (3% of V) contains 60% of M
- outer 50% of r (88% of V) contains 10% of M
- T rises fast towards the center (factor of 2500 for $1\,M_\odot$)
- $10\,M_\odot$: about 2 times the T's of the $1\,M_\odot$ star

stellar models

- behavior of ϵ follows T
- strong decrease of ϵ outwards from the center
- $1\,\mathrm{M}_\odot \rightarrow$
  ϵ drops by 2 dex between $m = 0$ and $m = 0.6$
- 90% of L is generated within $m/M \leq 0.3$
  $l/L \approx 0.99$ for $m/M = 0.5$
- central region of $10\,\mathrm{M}_\odot$ star $\rightarrow$ CNO cycle
- central region of $1\,\mathrm{M}_\odot$ star $\rightarrow$ pp chain

stellar models

- larger T dependence of CNO cycle $\rightarrow$ stronger concentration of ϵ in $10\,M_\odot$ star
- ϵ drops by 3 dex within $m/M = 0.3$ (only 1 dex for $1\,M_\odot$ star)
- $10\,M_\odot$ star further from the center
    - T lower $\rightarrow$ pp chain dominates
    - $\rightarrow$ slope same as for $1\,M_\odot$ star
- 90% of L is generated within 10% of M for the $10\,M_\odot$ star

Main Sequence

- sequence of chemically homogeneous models
- complete (hydrostatic and thermal) equilibrium
- identical H-rich composition
- central H-burning
- mass M varies along the sequence
- might be young stars that have just started H-burning after initial contraction
- stars change on nuclear time scales
- will "forget" their early history very quickly
  $\rightarrow$ complete equilibrium

Main Sequence !!

- $\tau_{\rm nuc}$ for H-burning very long $\rightarrow$ most observed stars are in this phase
- homogeneous models define the beginning this phase
- count stellar ages from start of H burning
- *Zero Age Main Sequence* (ZAMS)

Main Sequence

- simple relations on the MS!
- approximate stellar structure equations:
- use average values, .e.g, $R/2$, $M/2$, $L/2$
- derivatives: $dM_r/dr \rightarrow M/R$ etc
- assume $\kappa = \mathrm{const}$, $\epsilon \sim \rho T^\nu$
- gives reasonable results for similar ('homologous') objects

Main Sequence

- resulting relations:

$$\begin{aligned} \frac{M}{R} &\sim R^2\bar{\rho} \quad (1) \\ \frac{\bar{P}}{R} &\sim \frac{M}{R^2}\bar{\rho} \quad (2) \\ \frac{L}{R} &\sim R^2\bar{\rho}^2\bar{T}^{\nu} \quad (3) \\ \frac{\bar{T}}{R} &\sim \frac{\bar{\rho}L}{\bar{T}^3R^2} \quad (4) \end{aligned}$$

Main Sequence !!

- EOS: $\rho \sim P/T$ into (1) & (2) $\rightarrow$

$$\begin{aligned} \bar{\rho} &\sim M/R^3 \\ \bar{T} &\sim M/R \end{aligned}$$

- using this and (4) $\rightarrow$

$$L \sim M^3$$

Main Sequence

- solving (3) for R
$$R^3 \sim L\bar{\rho}^{-2}\bar{T}^{-\nu}$$
- replacing L, $\bar{\rho}$, and $\bar{T}$ $\rightarrow$
$$R^3 \sim M^3(M^{-2}R^6)(M^{-\nu}R^\nu)$$
- and thus
$$R \sim M^{\frac{\nu-1}{\nu+3}}$$
- exponent $= 0.75$ for $\nu = 13$, $= 0.5$ for $\nu = 5$
- assume
$$R \sim M^{3/4}$$

Main Sequence !!

- with

$$R \sim M^{3/4}$$

we obtain

$$L \sim R^4$$

- with $T_{\mathrm{eff}}^4 \sim L/R^2 \rightarrow$

$$T_{\mathrm{eff}}^4 \sim L^{1/2}$$

and therefore

$$L \sim T_{\mathrm{eff}}^8$$

- HRD $\rightarrow$ relation for the main sequence

$$\lg L = 8 \lg T_{\mathrm{eff}} + \mathrm{const.}$$

Main Sequence

- 'easy' to construct numerical models
- available for many sets of parameters and abundances
- here: $X = 0.685$, $Y = 0.294$, $Z = 0.021$

Model HRD MS

- HRD for stars with $0.1\,\mathrm{M}_\odot \le M \le 20\,\mathrm{M}_\odot$:

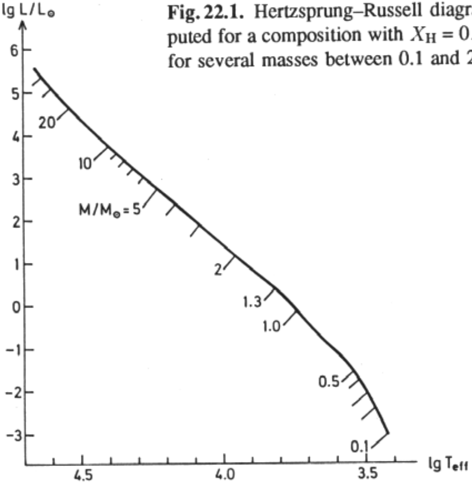


Fig. 22.1. Hertzsprung–Russell diagram with the zero-age main sequence computed for a composition with $X_\mathrm{H} = 0.685$, $X_\mathrm{He} = 0.294$. The locations of models for several masses between 0.1 and $22 M_\odot$ are indicated below the sequence

Main Sequence

- L and $T_{\rm eff}$ increase with M thus creating the ZAMS
- coincides more or less with observed MS

mass-radius relation

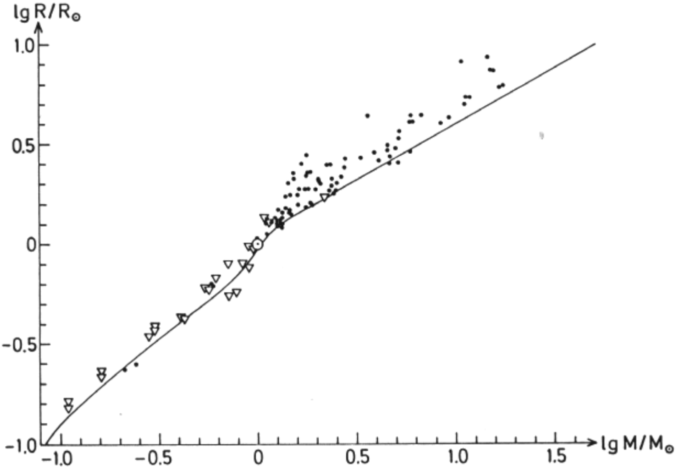


Fig. 22.2. The line shows the mass–radius relation for the models of the zero-age main sequence plotted in Fig. 22.1. For comparison, the best measurements (as selected by POPPER, 1980) of main-sequence components of detached (*dots*) and visual (*triangles*) binary systems are indicated

mass-radius relation

- R increases slowly, L increases strongly with M (see homology relations!)
- for interpolation over a range in M write:

$$R \propto M^{\xi}, \quad L \propto M^{\eta}$$

- upper mass range: $\xi = 0.57$
- lower mass range: $\xi = 0.8$

mass-radius relation

- pronounced max of the slope around $1\,M_\odot$
- remarkable deviation from homologous (self-similar) behavior
- decreasing T_{eff}
  - outer convective zones expand
  - $\rightarrow$ decreases R faster

M-L relation

- slope of M-L relation varies also with M
- average: $\eta \approx 3.2$
- $1\,\mathrm{M}_\odot \leq M \leq 10\,\mathrm{M}_\odot$ $\eta \approx 3.88$
- $1\,\mathrm{M}_\odot \leq M \leq 40\,\mathrm{M}_\odot$ $\eta \approx 3.35$
- decreasing slope due to increasing radiation pressure

Main Sequence

- quantitative tests of theory require precise measurements of R, M, L
- very difficult ...
- $M - R$ relation should be most reliable
- models are for ZAMS stars only!
- fit acceptable for a factor of 160 in M and 8 dex in L

White Dwarfs

- $M \approx 1\,\mathrm{M}_\odot$, $R \approx R_{\rm Earth}$
- $\rightarrow$ not normal stars
- physical conditions inside the WD:
  - assume constant density $\bar{\rho}$

$$\frac{dP}{dr} = -\frac{4}{3}\pi G \bar{\rho}^2 r$$

  - with $P(R) = 0$ this gives for the center

$$P_c \approx \frac{2}{3}\pi G \bar{\rho}^2 R^2 \approx 3.8 \times 10^{23}\,\mathrm{dyn/cm^2}$$

  - about 1.5×10^6 times larger than in the Sun!

White Dwarfs

- similar: central temperature

$$\frac{dT}{dr} \approx -\frac{3}{3ac}\frac{\kappa\rho}{T^3}\frac{L_r}{4\pi r^2}$$

- approximate

$$\frac{T_0 - T_c}{R} \approx -\frac{3}{3ac}\frac{\bar{\kappa}\bar{\rho}}{T_c^3}\frac{L}{4\pi R^2}$$

White Dwarfs

- $\bar{\kappa} \approx 0.2\,\mathrm{g/cm^2}$
- so that

$$T_c \approx \left[\frac{3\bar{\kappa}\bar{\rho}}{4ac}\frac{L}{4\pi R}\right]^{1/4} \approx 7.6 \times 10^7\,\mathrm{K}$$

- $\rightarrow$ no H left in the central regime!
- no nuclear energy source in WDs
- they just cool down ...

White Dwarfs

- end points of stars with $M < 8\,\mathrm{M}_\odot$
- consist mostly of C and O, could be ONeMg
- typical masses 0.4 to $0.7\,\mathrm{M}_\odot$
- supported by pressure from *degenerate electrons*

degenerate electron gas !!

- electrons are fermions (spin 1/2)
- $\rightarrow$ Pauli principle applies
- fermions will fill up quantum cells starting at the lowest energy
- limited volume $\rightarrow$ limited number of cells
- at 0 K all cells starting at the lowest energy are filled up
- $\rightarrow$ degenerate electron gas exerts pressure even at 0 K

degenerate electron gas !!

- completely degenerate:
- fermion gas filling all available quantum cells up to the
- Fermi energy

$$E_{\mathrm{F}} = \frac{\hbar}{2m}\left(3\pi^2 n_e\right)^{2/3}$$

degenerate electron gas

- express $E_{\rm F}$ with ρ and T:
- $n_e =$ (# of electrons/# of nucleons)(# of nucleons/volume)
- $n_e = (Z/A)(\rho/m_{\rm H})$
- therefore

$$E_{\rm F} = \frac{\hbar}{2m}\left[3\pi^2 \frac{Z}{A}\frac{\rho}{m_{\rm H}}\right]^{2/3}$$

degenerate electron gas

- average energy per electron $(3/2)kT$
- if this is $< E_{\mathrm{F}} \rightarrow$ no states $> E_{\mathrm{F}}$ are occupied
- therefore
$$\frac{3}{2}kT = \frac{\hbar}{2m}\left[3\pi^2\frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}\right]^{2/3}$$
for degenerate electron gases

degenerate electron pressure

- use Heisenberg $\Delta x \Delta p \approx \hbar$
- pressure-momentum relation:

$$P \approx \frac{1}{3} n_e p v$$

- $\Delta x = n_e^{-1/3}$ (minimal value!)
- therefore

$$p \approx \Delta p \approx \frac{\hbar}{\Delta x} = \hbar n_e^{1/3}$$

- so that

$$p \approx \hbar \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_{\mathrm{H}}} \right]^{1/3}$$

degenerate electron pressure

- velocity v for non-relat. electrons:

$$v = \frac{p}{m_e} \approx \frac{\hbar}{m_e} n_e^{1/3} \approx \frac{\hbar}{m_e} \left[\frac{Z}{A} \frac{\rho}{m_{\mathrm{H}}} \right]^{1/3}$$

- inserting all these relations $\rightarrow$

$$P \approx \frac{1}{3} \frac{\hbar^2}{m_e} \left[\frac{Z}{A} \frac{\rho}{m_{\mathrm{H}}} \right]^{5/3}$$

- more accurate derivation

$$\frac{1}{3} \rightarrow \frac{(3\pi^2)^{2/3}}{5}$$

M-R relationship !!

- set central pressure P_c equal to electron degeneracy pressure

$$\frac{2}{3}\pi G\bar{\rho}^2 R_{\mathrm{WD}}^2 = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}\right]^{5/3}$$

- $\bar{\rho} = M_{\mathrm{WD}}/(4/3)\pi R_{\mathrm{WD}}^3$
- therefore

$$R_{\mathrm{WD}} \approx \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{G m_e M_{\mathrm{WD}}^{1/3}}\left[\frac{Z}{A}\frac{1}{m_{\mathrm{H}}}\right]^{5/3}$$

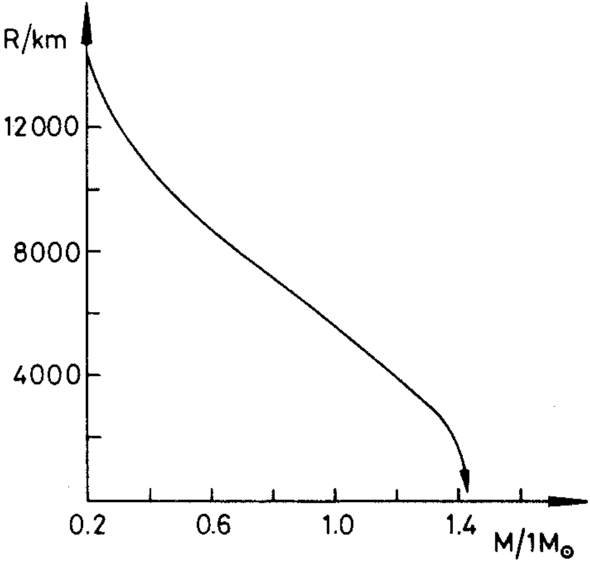
M-R relationship !!

- important:

$$M_{\mathrm{WD}} R_{\mathrm{WD}}^3 = \mathrm{const.}$$

- Mass-Radius relation for WDs!

M-*R* relationship



Chandrasekhar limit !!

- more massive white dwarfs are *smaller*!
- simple $M - R$ relation is not valid for $M \to \infty$:
- speed of the electrons in WDs:

$$v \approx \frac{\hbar}{m_e} \left[\frac{Z}{A} \frac{\rho}{m_{\mathrm{H}}} \right]^{1/3} \approx 1/3c$$

- v cannot exceed c
- relativistic treatment reduces v
- $\to$ smaller pressures
- $M - R$ relation changes $\to$ zero radius at finite mass
- $\to$ *Chandrasekhar limit*

Chandrasekhar limit

- estimate: set $v = c$
- non-relativistic limit

$$P \approx \frac{(3\pi^2)^{2/3}}{5}\hbar c \left[\frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}\right]^{4/3}$$

- NB: this corresponds to $P \sim \rho^{4/3}$ (dynamically unstable)

Chandrasekhar limit !!

- this gives a maximum central pressure P_c and limiting mass

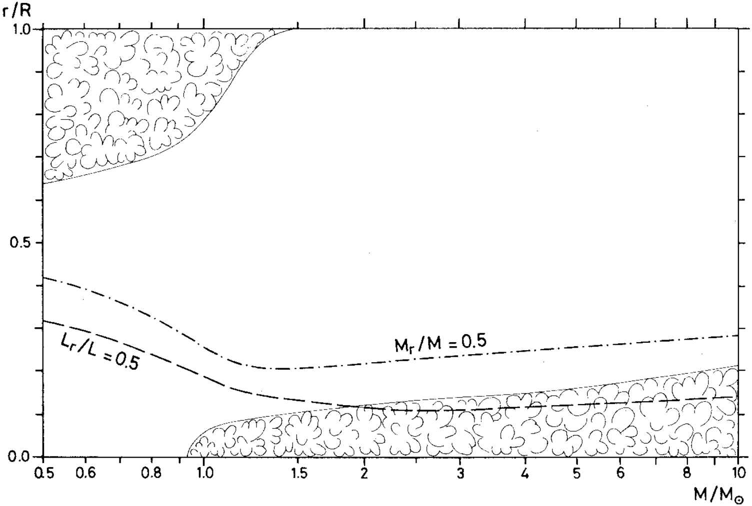
$$M_{\mathrm{Ch}} \approx \frac{3\sqrt{2\pi}}{8} \left(\frac{\hbar c}{G}\right)^{2/3} \left[\frac{Z}{A}\frac{1}{m_{\mathrm{H}}}\right]^2 \approx 0.44\,\mathrm{M}_\odot$$

- $\rightarrow$ Chandrasekhar limit
- correct value: $M_{\mathrm{Ch}} = 1.44\,\mathrm{M}_\odot$
- depends on composition!
- T doesn't even show up!

Convection !!

- convection is 'active' if it delivers more effective energy transport
- $\rightarrow$ Schwarzschild criterion
- happens mostly in
  - ionization zones (adiabatic gradient reduced)
  - high opacity (χ large)
  - large radiation flux F

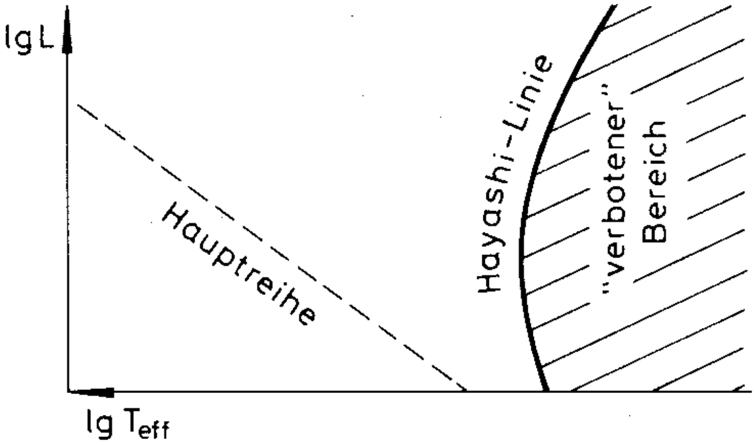
Convection in stars



Convection

- convective envelope:
  - low mass stars
  - for $M < 0.2\,\mathrm{M}_\odot \rightarrow$ MS star *fully convective!*
  - produced for *all* cool envelopes!
  - depth of convection zone depends on M, L and T_{eff}
  - $\rightarrow$ *Hayashi line*

Hayashi line



Hayashi line

- is function of stellar mass!
- fully convective stars are on the HL for their mass
- 'left' of the HL: dynamically stable
- 'right' of the HL: dynamically unstable
    - stars must contract or expand on dynamical timescales
    - cannot cross HL during evolution