

Einführung in die Astronomie II

Teil 8

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Übersicht Teil 8

- ▶ stellar structure
 - ▶ stellar models
 - ▶ the main sequence
 - ▶ white dwarfs
 - ▶ convection
 - ▶ Hayashi line

stellar models

- ▶ conservation of mass

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

- ▶ hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho$$

stellar models

- ▶ conservation of energy

$$L_r = \int_0^{M_r} \epsilon dM_r = \int_0^r \epsilon 4\pi r^2 \rho dr$$

- ▶ energy transport (diffusion)

$$\frac{dT}{dr} = -\frac{3}{64\pi} \frac{\chi}{\sigma} \frac{L_r}{r^2 T^3}$$

- ▶ 4 equations for 4 unknowns!

stellar models

- ▶ mixed boundary conditions
 - ▶ $r = 0$
 - ▶ $M_r = 0$
 - ▶ $L_r = 0$
 - ▶ $r = R$
 - ▶ $T \rightarrow 0$
 - ▶ $P \rightarrow 0$
- ▶ solution \rightarrow numerical!

stellar models

- ▶ characteristic variables as functions of m/M :

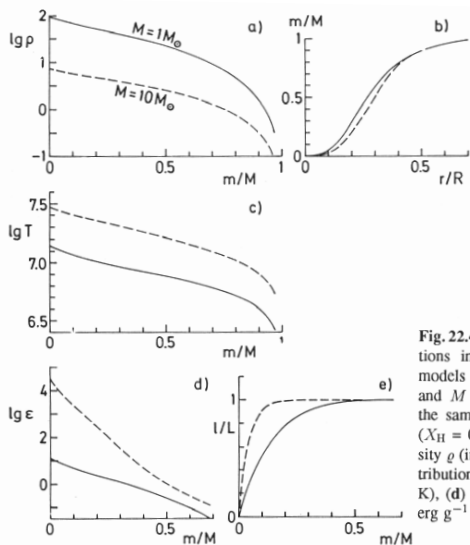


Fig. 22.4. a, b The run of some functions inside zero-age main-sequence models for $M = 1M_{\odot}$ (solid lines) and $M = 10M_{\odot}$ (dashed lines) with the same composition as in Fig. 22.1 ($X_{\text{H}} = 0.685$, $X_{\text{He}} = 0.294$); (a) density ρ (in g cm^{-3}), (b) radial mass distribution $m(r)$, (c) temperature T (in K), (d) nuclear energy production (in $\text{erg g}^{-1} \text{s}^{-1}$), (e) local luminosity l

stellar models

- ▶ density rises steeply towards center (9 dex from atmosphere values)
- ▶ $10 M_{\odot}$ model has 1 dex *lower* central density!
- ▶ very strong concentration of mass towards the center of the stars
- ▶ $1 M_{\odot}$: inner 30% of r (3% of V) contains 60% of M
- ▶ outer 50% of r (88% of V) contains 10% of M
- ▶ T rises fast towards the center
factor of 2500 for $1 M_{\odot}$)
- ▶ $10 M_{\odot}$: about 2 times the T 's of the $1 M_{\odot}$ star

stellar models

- ▶ behavior of ϵ follows T
- ▶ strong decrease of ϵ outwards from the center
- ▶ $1 M_{\odot} \rightarrow$
 ϵ drops by 2 dex between $m = 0$ and $m = 0.6$
- ▶ 90% of L is generated within $m/M \leq 0.3$
 $l/L \approx 0.99$ for $m/M = 0.5$
- ▶ central part of $10 M_{\odot}$ star \rightarrow CNO cycle
- ▶ central part of $1 M_{\odot}$ star \rightarrow pp chain

stellar models

- ▶ larger T dependence of CNO cycle \rightarrow stronger concentration of ϵ in $10 M_{\odot}$ star
- ▶ ϵ drops by 3 dex within $m/M = 0.3$ (only 1 dex for $1 M_{\odot}$ star)
- ▶ $10 M_{\odot}$ star further from the center
 - ▶ T lower \rightarrow pp chain dominates
 - ▶ \rightarrow slope same as for $1 M_{\odot}$ star
- ▶ 90% of L is generated within 10% of M for the $10 M_{\odot}$ star

Main Sequence

- ▶ sequence of chemically homogeneous models
- ▶ complete (hydrostatic and thermal) equilibrium
- ▶ identical H-rich composition
- ▶ central H-burning
- ▶ mass M varies along the sequence
- ▶ might be young stars that have just started H-burning after initial contraction
- ▶ stars change on nuclear time scales
- ▶ will “forget” their early history very quickly
→ complete equilibrium

Main Sequence !!

- ▶ τ_{nuc} for H-burning very long \rightarrow most observed stars are in this phase
- ▶ homogeneous models define the beginning this phase
- ▶ count stellar ages from start of H burning
- ▶ *Zero Age Main Sequence* (ZAMS)

Main Sequence

- ▶ simple relations on the MS!
- ▶ approximate stellar structure equations:
- ▶ use average values, .e.g, $R/2$, $M/2$, $L/2$
- ▶ derivatives: $dM_r/dr \rightarrow M/R$ etc
- ▶ assume $\kappa = \text{const}$, $\epsilon \sim \rho T^\nu$
- ▶ gives reasonable results for similar ('homologous') objects

Main Sequence

- ▶ resulting relations:

$$\frac{M}{R} \sim R^2 \bar{\rho} \quad (1)$$

$$\frac{\bar{P}}{R} \sim \frac{M}{R^2} \bar{\rho} \quad (2)$$

$$\frac{L}{R} \sim R^2 \bar{\rho}^2 \bar{T}^\nu \quad (3)$$

$$\frac{\bar{T}}{R} \sim \frac{\bar{\rho} L}{\bar{T}^3 R^2} \quad (4)$$

Main Sequence !!

- ▶ EOS: $\rho \sim P/T$ into (1) & (2) \rightarrow

$$\bar{\rho} \sim M/R^3$$

$$\bar{T} \sim M/R$$

- ▶ using this and (4) \rightarrow

$$L \sim M^3$$

Main Sequence

- ▶ solving (3) for R

$$R^3 \sim L \bar{\rho}^{-2} \bar{T}^{-\nu}$$

- ▶ replacing L , $\bar{\rho}$, and $\bar{T} \rightarrow$

$$R^3 \sim M^3 (M^{-2} R^6) (M^{-\nu} R^{\nu})$$

- ▶ and thus

$$R \sim M^{\frac{\nu-1}{\nu+3}}$$

- ▶ exponent = 0.75 for $\nu = 13$, = 0.5 for $\nu = 5$
- ▶ assume

$$R \sim M^{3/4}$$

Main Sequence !!

- ▶ with

$$R \sim M^{3/4}$$

we obtain

$$L \sim R^4$$

- ▶ with $T_{\text{eff}}^4 \sim L/R^2 \rightarrow$

$$T_{\text{eff}}^4 \sim L^{1/2}$$

and therefore

$$L \sim T_{\text{eff}}^8$$

- ▶ HRD \rightarrow relation for the main sequence

$$\lg L = 8 \lg T_{\text{eff}} + \text{const.}$$

Main Sequence

- ▶ 'easy' to construct numerical models
- ▶ available for many sets of parameters and abundances
- ▶ here: $X = 0.685$, $Y = 0.294$, $Z = 0.021$

Model HRD MS

- ▶ HRD for stars with $0.1 M_{\odot} \leq M \leq 20 M_{\odot}$:

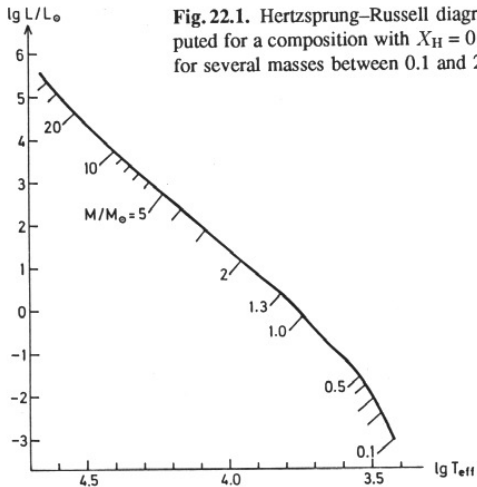


Fig. 22.1. Hertzsprung–Russell diagram with the zero-age main sequence computed for a composition with $X_{\text{H}} = 0.685$, $X_{\text{He}} = 0.294$. The locations of models for several masses between $0.1 M_{\odot}$ and $22 M_{\odot}$ are indicated below the sequence

Main Sequence

- ▶ L and T_{eff} increase with M thus creating the ZAMS
- ▶ coincides more or less with observed MS

mass-radius relation

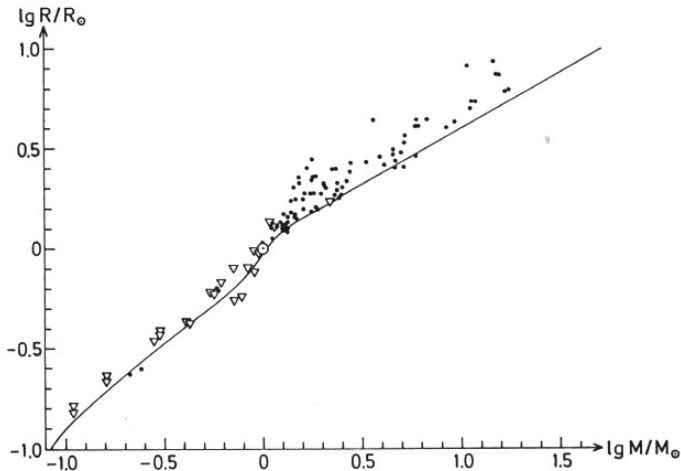


Fig. 22.2. The line shows the mass-radius relation for the models of the zero-age main sequence plotted in Fig. 22.1. For comparison, the best measurements (as selected by POPPER, 1980) of main-sequence components of detached (*dots*) and visual (*triangles*) binary systems are indicated

mass-radius relation

- ▶ R increases slowly, L increases strongly with M (see homology relations!)
- ▶ for interpolation over a range in M write:

$$R \propto M^\xi, \quad L \propto M^\eta$$

- ▶ upper mass range: $\xi = 0.57$
- ▶ lower mass range: $\xi = 0.8$

mass-radius relation

- ▶ pronounced max of the slope around $1 M_{\odot}$
- ▶ remarkable deviation from homologous (self-similar) behavior
- ▶ decreasing T_{eff}
 - ▶ outer convective zones expand
 - ▶ \rightarrow decreases R faster

M - L relation

- ▶ slope of M - L relation varies also with M
- ▶ average: $\eta \approx 3.2$
- ▶ $1 M_{\odot} \leq M \leq 10 M_{\odot}$ $\eta \approx 3.88$
- ▶ $1 M_{\odot} \leq M \leq 40 M_{\odot}$ $\eta \approx 3.35$
- ▶ decreasing slope due to increasing radiation pressure

Main Sequence

- ▶ quantitative tests of theory require precise measurements of R , M , L
- ▶ very difficult . . .
- ▶ $M - R$ relation should be most reliable
- ▶ models are for ZAMS stars only!
- ▶ fit acceptable for a factor of 160 in M and 8 dex in L

White Dwarfs

- ▶ $M \approx 1 M_{\odot}$, $R \approx R_{\text{Earth}}$
- ▶ → not normal stars
- ▶ physical conditions inside the WD:
 - ▶ assume constant density $\bar{\rho}$

$$\frac{dP}{dr} = -\frac{4}{3}\pi G\bar{\rho}^2 r$$

- ▶ with $P(R) = 0$ this gives for the center

$$P_c \approx \frac{2}{3}\pi G\bar{\rho}^2 R^2 \approx 3.8 \times 10^{23} \text{ dyn/cm}^2$$

- ▶ about 1.5×10^6 times larger than in the Sun!

White Dwarfs

- ▶ similar: central temperature

$$\frac{dT}{dr} \approx -\frac{3}{3ac} \frac{\kappa\rho}{T^3} \frac{L_r}{4\pi r^2}$$

- ▶ approximate

$$\frac{T_0 - T_c}{R} \approx -\frac{3}{3ac} \frac{\bar{\kappa}\bar{\rho}}{T_c^3} \frac{L}{4\pi R^2}$$

White Dwarfs

- ▶ $\bar{\kappa} \approx 0.2 \text{ g/cm}^2$
- ▶ so that

$$T_c \approx \left[\frac{3\bar{\kappa}\bar{\rho}}{4ac} \frac{L}{4\pi R} \right]^{1/4} \approx 7.6 \times 10^7 \text{ K}$$

- ▶ → no H left in the central regime!
- ▶ no nuclear energy source in WDs
- ▶ they just cool down ...

White Dwarfs

- ▶ end points of stars with $M < 8 M_{\odot}$
- ▶ consist mostly of C and O, could be ONeMg
- ▶ typical masses 0.4 to 0.7 M_{\odot}
- ▶ supported by pressure from *degenerate electrons*

degenerate electron gas !!

- ▶ electrons are fermions (spin $1/2$)
- ▶ → Pauli principle applies
- ▶ fermions will fill up quantum cells starting at the lowest energy
- ▶ limited volume → limited number of cells
- ▶ at 0 K all cells starting at the lowest energy are filled up
- ▶ → degenerate electron gas exerts pressure even at 0 K

degenerate electron gas !!

- ▶ completely degenerate:
- ▶ fermion gas filling all available quantum cells up to the
- ▶ Fermi energy

$$E_F = \frac{\hbar^2}{2m} (3\pi^2 n_e)^{2/3}$$

degenerate electron gas

- ▶ express E_F with ρ and T :
- ▶ $n_e =$
(# of electrons/# of nucleons)(# of nucleons/volume)
- ▶ $n_e = (Z/A)(\rho/m_H)$
- ▶ therefore

$$E_F = \frac{\hbar}{2m} \left[3\pi^2 \frac{Z}{A} \frac{\rho}{m_H} \right]^{2/3}$$

degenerate electron gas

- ▶ average energy per electron $(3/2)kT$
- ▶ if this is $< E_F \rightarrow$ no states $> E_F$ are occupied
- ▶ therefore

$$\frac{3}{2}kT = \frac{\hbar}{2m} \left[3\pi^2 \frac{Z}{A} \frac{\rho}{m_H} \right]^{2/3}$$

for degenerate electron gases

degenerate electron pressure

- ▶ use Heisenberg $\Delta x \Delta p \approx \hbar$
- ▶ pressure-momentum relation:

$$P \approx \frac{1}{3} n_e p v$$

- ▶ $\Delta x = n_e^{-1/3}$ (minimal value!)
- ▶ therefore

$$p \approx \Delta p \approx \frac{\hbar}{\Delta x} = \hbar n_e^{1/3}$$

- ▶ so that

$$p \approx \hbar \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_{\text{H}}} \right]^{1/3}$$

degenerate electron pressure

- ▶ velocity v for non-relat. electrons:

$$v = \frac{p}{m_e} \approx \frac{\hbar}{m_e} n_e^{1/3} \approx \frac{\hbar}{m_e} \left[\frac{Z}{A} \frac{\rho}{m_H} \right]^{1/3}$$

- ▶ inserting all these relations \rightarrow

$$P \approx \frac{1}{3} \frac{\hbar^2}{m_e} \left[\frac{Z}{A} \frac{\rho}{m_H} \right]^{5/3}$$

- ▶ more accurate derivation

$$\frac{1}{3} \rightarrow \frac{(3\pi^2)^{2/3}}{5}$$

M - R relationship !!

- ▶ set central pressure P_c equal to electron degeneracy pressure

$$\frac{2}{3}\pi G \bar{\rho}^2 R_{\text{WD}}^2 = \frac{(3\pi^2)^{2/3} \hbar^2}{5 m_e} \left[\frac{Z}{A} \frac{\rho}{m_{\text{H}}} \right]^{5/3}$$

- ▶ $\bar{\rho} = M_{\text{WD}} / (4/3)\pi R_{\text{WD}}^3$
- ▶ therefore

$$R_{\text{WD}} \approx \frac{(18\pi)^{2/3}}{10} \frac{\hbar^2}{G m_e M_{\text{WD}}^{1/3}} \left[\frac{Z}{A} \frac{1}{m_{\text{H}}} \right]^{5/3}$$

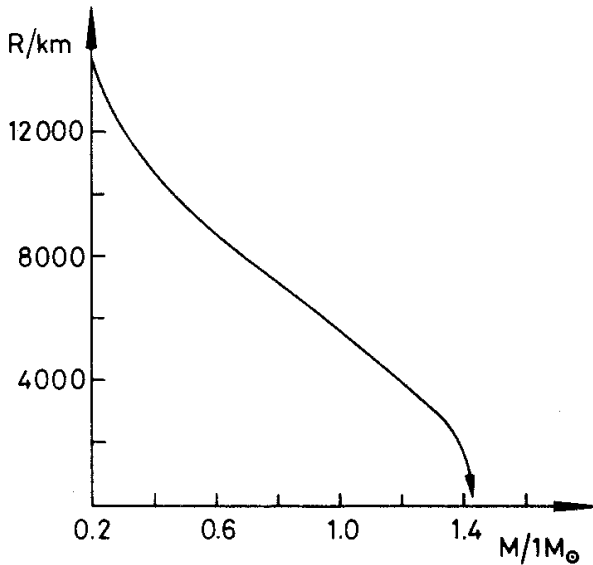
M - R relationship !!

- ▶ important:

$$M_{\text{WD}} R_{\text{WD}}^3 = \text{const.}$$

- ▶ Mass-Radius relation for WDs!

M - R relationship



Chandrasekhar limit !!

- ▶ more massive white dwarfs are *smaller*!
- ▶ simple $M - R$ relation is not valid for $M \rightarrow \infty$:
- ▶ speed of the electrons in WDs:

$$v \approx \frac{\hbar}{m_e} \left[\frac{Z}{A} \frac{\rho}{m_H} \right]^{1/3} \approx 1/3c$$

- ▶ v cannot exceed c
- ▶ relativistic treatment reduces v
- ▶ \rightarrow smaller pressures
- ▶ $M - R$ relation changes \rightarrow zero radius at finite mass
- ▶ \rightarrow *Chandrasekhar limit*

Chandrasekhar limit

- ▶ estimate: set $v = c$
- ▶ non-relativistic limit

$$P \approx \frac{(3\pi^2)^{2/3}}{5} \hbar c \left[\frac{Z}{A} \frac{\rho}{m_{\text{H}}} \right]^{4/3}$$

- ▶ NB: this corresponds to $P \sim \rho^{4/3}$ (dynamically unstable)

Chandrasekhar limit !!

- ▶ this gives a maximum central pressure P_c and limiting mass

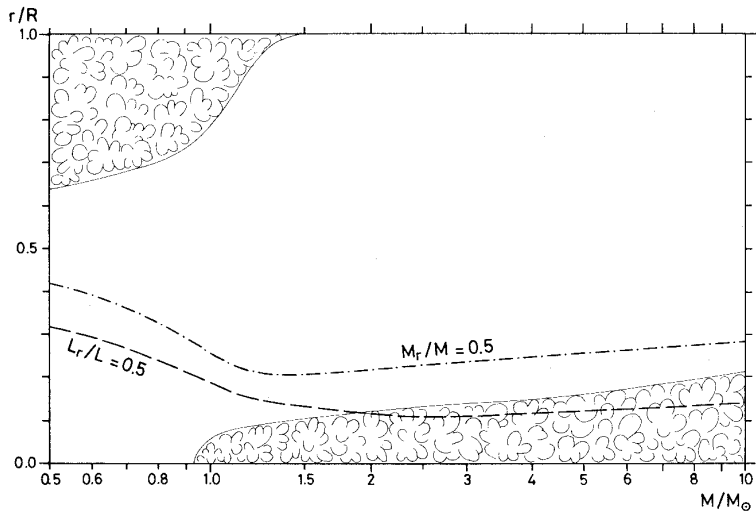
$$M_{\text{Ch}} \approx \frac{3\sqrt{2\pi}}{8} \left(\frac{\hbar c}{G} \right)^{2/3} \left[\frac{Z}{A} \frac{1}{m_{\text{H}}} \right]^2 \approx 0.44 M_{\odot}$$

- ▶ → Chandrasekhar limit
- ▶ correct value: $M_{\text{Ch}} = 1.44 M_{\odot}$
- ▶ depends on composition!
- ▶ T doesn't even show up!

Convection !!

- ▶ convection is 'active' if it delivers more effective energy transport
- ▶ → Schwarzschild criterion
- ▶ happens mostly in
 - ▶ ionization zones (adiabatic gradient reduced)
 - ▶ high opacity (χ large)
 - ▶ large radiation flux F

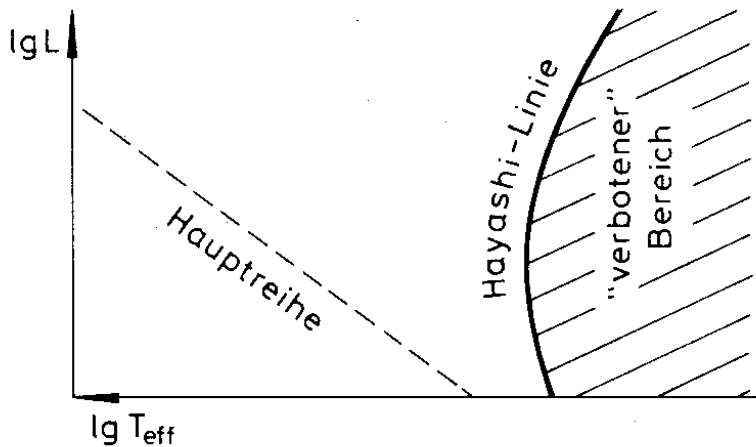
Convection in stars



Convection

- ▶ convective envelope:
 - ▶ low mass stars
 - ▶ for $M < 0.2 M_{\odot}$ → MS star *fully convective!*
 - ▶ produced for *all* cool envelopes!
 - ▶ depth of convection zone depends on M , L and T_{eff}
 - ▶ → *Hayashi line*

Hayashi line



Hayashi line

- ▶ is function of stellar mass!
- ▶ fully convective stars are on the HL for their mass
- ▶ 'left' of the HL: dynamically stable
- ▶ 'right' of the HL: dynamically unstable
 - ▶ stars must contract or expand on dynamical timescales
 - ▶ cannot cross HL during evolution