Stellar/Planetary Atmospheres

Part 13: convection

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Topics

- Convection
  - Schwarzschild criterion
  - mixing length theory
  - T gradient in convective atmospheres

Convection

- important for cool stars
- M dwarfs: fully convective starting outer atmosphere
- no real theory of convective energy transport!
- comparatively simplistic model for convection
- especially bad in optically thin media!

Schwarzschild criterion

- is volume element stable against small displacements?
- index $E \rightarrow$ volume element
- assume adiabatic motion (no energy exchange)
- element in pressure equilibrium with surroundings

$$(\Delta\rho)_E = \left(\frac{d\rho}{dr}\right)_{\mathrm{ad}} \Delta r$$

- will continue to rise if $\rho(R)$ is larger than ρ_E, i.e.

$$(\Delta\rho)_E < (\Delta\rho)_R = \left(\frac{d\rho}{dr}\right)_R dr$$

Schwarzschild criterion

- hydrostatic atmosphere $\rightarrow$

$$\frac{d\rho}{dr} < 0$$

so that

$$\left|\left(\frac{d\rho}{dr}\right)_{\mathrm{ad}}\right| < \left|\left(\frac{d\rho}{dr}\right)_{R}\right|$$

for convection

Schwarzschild criterion

- in this form not very practical
- use EOS to simplify:
- element: adiabate of perfect gas $\rightarrow$

$$\ln P = \gamma \ln \rho + C$$

so that

$$\left(\frac{d \ln \rho}{dr}\right)_{\mathrm{ad}} = \frac{1}{\gamma} \left(\frac{d \ln P}{dr}\right)_{\mathrm{ad}}$$

Schwarzschild criterion

- atmospheric gas not perfect:
  - radiation pressure
  - ionization
  - dissociation

Schwarzschild criterion

- $\rightarrow$ adiabate more complicated:

$$\nabla_{\mathrm{ad}} = \left(\frac{d \ln T}{d \ln P}\right)_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}$$

  with

$$\frac{dP}{P} = \frac{\Gamma_2}{\Gamma_2 - 1}\frac{dT}{T}$$

  as definition of Γ_2 along adiabate

- perfect gas $\rightarrow$

$$\Gamma_2 = \gamma = \frac{C_p}{C_V}$$

- ∇_{ad} has to be calculated with the EOS!

Schwarzschild criterion

- in the surroundings (not adiabatic) $\rightarrow$

$$\ln P = \ln \rho + \ln T + C$$

so that

$$\left(\frac{d \ln \rho}{dr}\right)_R = \left(\frac{d \ln P}{dr}\right)_R - \left(\frac{d \ln T}{dr}\right)_R$$

Schwarzschild criterion

- inserting into stability criterion $\rightarrow$

$$\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{-d\ln P}{dr}\right)_R < \left(\frac{-d\ln T}{dr}\right)_R$$

  (due to pressure equilibrium)
- so that

$$\left(\frac{d\ln T}{d\ln P}\right)_R > \left(\frac{\gamma-1}{\gamma}\right) = \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{ad}}$$

Schwarzschild criterion

- with

$$\nabla = \left(\frac{d \ln T}{d \ln P}\right)$$

this gives a useful form of the

- Schwarzschild instability criterion

$$\nabla_R > \nabla_{\mathrm{ad}}$$

examples

- perfect, mon-atomic gas:

$$\gamma = \frac{C_p}{C_V} = \frac{5}{3}$$

  $\rightarrow$ convective if $\nabla_R > 0.4$

- pure radiation pressure:

$$\Gamma_2 = \frac{4}{3}$$

  $\rightarrow$ convective if $\nabla_R > 0.25$

examples

- ionizing hydrogen:

$$\Gamma_2 \approx 1.1$$

$\rightarrow$ convective if $\nabla_R > 0.091 \approx 0.1$

convection

- radiative temperature gradient for diffusion:

$$\frac{dT}{dr} = -\left(\frac{3\pi}{16\sigma}\right)\frac{\rho\kappa_{\mathrm{Ross}}F}{T^3}$$

- so that

$$\nabla_R = \left(\frac{3\pi}{16\sigma}\right)\frac{\kappa_{\mathrm{Ross}}P}{T^4 g}F$$

convection

- large opacities $\rightarrow$ large T-gradients for given F
- opacities are large in $P - T$ regions with ionization
- $\rightarrow \nabla_{\mathrm{ad}}$ is small, too!
- $\rightarrow$ convection occurs frequently
- often: H ionization zones

mixing length theory

- originally developed by Prandtl (1925)
- adapted to astrophysics by L. Biermann
- modern form by Böhm-Vitense (1950's)
- basic idea:
  - energy transport by bubbles
  - move up (hot) and down (cool) through the atmosphere
  - bubbles carry excess energy upward
  - dissolve after traveling characteristic distance
  - $\rightarrow$ *mixing length*
  - then dump excess energy into surroundings
  - analogous for downward moving bubbles!

definitions

- ∇_R: radiative T-gradient if no convection
- ∇_{ad}: adiabatic T-gradient
- ∇_E: T-gradient for individual convection element (bubble)
- ∇: final T-gradient in the surrounding gas with both radiative and convective energy transport

MLT

- in general:

$$\nabla_R \geq \nabla \geq \nabla_E \geq \nabla_{\mathrm{ad}}$$

- rising element (pressure equilibrium) $\rightarrow$
- temperature difference δT to surrounding gas
- $\rightarrow$ excess bubble energy content $\rho C_p \delta T$
- δT from T-gradients and traveled distance Δr

$$\delta T = \left[\left(\frac{-dT}{dr} \right) - \left(\frac{-dT}{dr} \right)_E \right] \Delta r$$

MLT

- elements move with average velocity $\bar{v}$
- $\rightarrow$ convective flux $F_{\mathrm{conv}} \rightarrow$

$$F_{\mathrm{conv}} = \rho C_P \bar{v} \left[\left(\frac{-dT}{dr} \right) - \left(\frac{-dT}{dr} \right)_E \right] \Delta r$$

- elements travel a distance $\ell \rightarrow$ *mixing length*
- on the average, each layer of the atmosphere is crossed by bubbles that traveled $\ell/2 = \Delta r$

MLT

- hydrostatic equation:

$$dP/dr = -\rho g$$

- with $\nabla = d \ln T / d \ln P \rightarrow$

$$F_{\rm conv} = \frac{1}{2} \frac{g \rho^2 C_P T \bar{v} \ell}{P} (\nabla - \nabla_E)$$

MLT

- simplify with pressure scale height H

$$\frac{1}{H} = -\frac{d \ln P}{dr} = \frac{\rho g}{P}$$

- this gives

$$F_{\mathrm{conv}} = \frac{1}{2}\rho C_P \bar{v} T \left(\frac{\ell}{H}\right)(\nabla - \nabla_E)$$

- now we need to compute $\bar{v}$!

convective velocity $\bar{v}$

- assumption:
  - bubble mean kinetic energy $W =$ work done by buoyant forces

$$f_B = -g\delta\rho$$

- f_B: buoyant forces
- $\delta\rho$: density difference bubble – surroundings

convective velocity $\bar{v}$

- EOS $\rightarrow$

$$\rho = \mu \frac{P}{T}$$

- $\rightarrow$

$$\ln \rho = \ln P - \ln T + \ln \mu$$

so that

$$\begin{aligned} \frac{d\rho}{\rho} &= \frac{dP}{P} - \frac{dT}{T} + \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P \frac{dT}{T} \\ &= \frac{dP}{P} - Q\left(\frac{dT}{T}\right) \end{aligned}$$

convective velocity $\bar{v}$

- with
$$Q = 1 - \left(\frac{\partial \ln \mu}{\partial \ln T} \right)_P$$
- pressure equilibrium $\rightarrow \delta P = dP = 0$
- with $\delta T = (dT/dr)\Delta r \rightarrow$

$$f_B = \frac{gQ\rho}{T}\delta T = \frac{gQ\rho}{T}\left[\left(\frac{-dT}{dr}\right) - \left(\frac{-dT}{dr}\right)_E\right]\Delta r$$

convective velocity $\bar{v}$

- work done by f_B for an bubble moving distance Δ

$$W = \int_0^{\Delta} f_B(\Delta r)\, d\Delta r = \frac{1}{2}\frac{gQ\rho}{T}\left[\left(\frac{-dT}{dr}\right) - \left(\frac{-dT}{dr}\right)_E\right]\Delta^2$$

- averaging over all bubbles in a layer
- $\rightarrow$ set $\Delta = \ell/2 \rightarrow$

$$W = \frac{1}{8}\left(\rho g Q H\right)\left(\nabla - \nabla_E\right)\left(\frac{\ell}{H}\right)^2$$

convective velocity $\bar{v}$

- $\approx W/2$ lost by friction (pushing gas away)
- $\rightarrow$ kinetic energy of the bubble

$$\frac{1}{2}\rho\bar{v}^2 = \frac{1}{2}W$$

and thus

$$\bar{v} = \frac{1}{2\sqrt{2}}\sqrt{gQH}\,(\nabla - \nabla_E)^{1/2}\left(\frac{\ell}{H}\right)$$

- inserting $\rightarrow$

$$F_{\mathrm{conv}} = \frac{1}{4\sqrt{2}}\sqrt{gQH}\rho C_P T\,(\nabla - \nabla_E)^{3/2}\left(\frac{\ell}{H}\right)^2$$

MLT

- fundamental problem: ℓ
- typically parameterized as fraction of H
- range: $0.5 \leq \ell/H \leq 2$
- example sun: $\ell/H \approx 1.5$
- other prescription (non-local averages etc) cannot overcome the fundamental problem

MLT

- upward moving bubble $\rightarrow$
  - bubble hotter than surrounding gas
  - $\rightarrow$ energy loss through radiation
  - $\rightarrow$ reduces the energy transported by convection
- introduce convective efficiency γ
- ratio of excess energy at time of dissolution to energy lost by radiation over bubble lifetime $\rightarrow$

$$\gamma = \frac{(\nabla - \nabla_E)}{(\nabla - \nabla_{\mathrm{ad}}) - (\nabla - \nabla_E)} = \frac{\nabla - \nabla_E}{\nabla_E - \nabla_{\mathrm{ad}}}$$

MLT

- excess energy set free by the bubble:

$$\rho C_P V \delta T$$

- V: bubble volume
- losses due to radiation depend on optical thickness of the bubble

MLT

- optically thick elements:
- radiation losses in diffusion approximation

$$\frac{16\sigma T^3}{3\rho\kappa_{\rm Ross}}\frac{dT}{dr} \approx \frac{16\sigma T^3}{3\rho\kappa_{\rm Ross}}\frac{\delta T}{\ell}$$

MLT

- bubble with surface area A and lifetime $\ell/\bar{v} \rightarrow$
- total energy lost:

$$\frac{16\sigma T^3}{3\rho\kappa_{\rm Ross}}\frac{A}{\bar{v}}\delta T$$

- $\rightarrow$ convective efficiency

$$\gamma_{\rm thick} = \frac{3\rho C_P \bar{v}}{16\sigma T^3}\kappa_{\rm Ross}\rho\left(\frac{V}{A}\right)$$

MLT

- choice of V/A is largely arbitrary ...
- spherical element with radius ℓ:

$$\frac{V}{A} \rightarrow \frac{\ell}{3}$$

- so that with $\tau_\ell = \kappa_{\mathrm{Ross}} \rho \ell \gg 1$

$$\gamma_{\mathrm{thick}} = \frac{\rho C_P \bar{v}}{16 \sigma T^3} \tau_\ell$$

MLT

- optically thin bubbles:
- energy loss of bubble with volume V and lifetime $\ell/\bar{v}$ and average $\delta T/2 \rightarrow$

$$4\pi\left(\frac{4\sigma T^3}{\pi}\right)\frac{\delta T}{2}\left(V\kappa_{\mathrm{Ross}}\rho\right)\frac{\ell}{\bar{v}}$$

- so that

$$\gamma_{\mathrm{thin}} = \frac{C_P\bar{v}}{8\sigma T^3\kappa_{\mathrm{Ross}}\ell} = \frac{\rho C_P\bar{v}}{8\sigma T^3}\frac{1}{\tau_\ell}$$

- to account for both limits $\rightarrow$ linear interpolation

$$\gamma = \frac{\rho C_P\bar{v}}{8\sigma T^3}\frac{1+\frac{1}{2}\tau_\ell^2}{\tau_\ell}$$

MLT

- combine with expression for $\bar{v}$ and

$$\gamma = \frac{\nabla - \nabla_E}{\nabla_E - \nabla_{\rm ad}}$$

to

$$\frac{\nabla_E - \nabla_{\rm ad}}{(\nabla - \nabla_E)^{1/2}} = \frac{16\sqrt{2}\sigma T^3}{\rho C_P \sqrt{gQH}(\ell/H)} \frac{\tau_\ell}{1 + \frac{1}{2}\tau_\ell^2} \equiv B$$

- use this to compute the real T-gradient ∇ in a convective atmosphere!

computing ∇

- energy conservation:

$$F_{\mathrm{rad}} + F_{\mathrm{conv}} = \sigma T_{\mathrm{eff}}^4$$

- T-gradient ∇ constrained by

$$\nabla_R \geq \nabla \geq \nabla_{\mathrm{ad}}$$

computing ∇

- diffusion approximation $\rightarrow$

$$\frac{F_{\mathrm{rad}}}{F} = \frac{\nabla}{\nabla_R}$$

so that

$$F_{\mathrm{conv}} = \sigma T_{\mathrm{eff}}^4 \left(1 - \frac{\nabla}{\nabla_R}\right)$$

computing ∇

- insert expression for $F_{\rm conv}$ $\rightarrow$

$$A\left(\nabla - \nabla_E\right)^{3/2} = \nabla_R - \nabla$$

with

$$A \equiv \frac{\nabla_R(\ell/H)^2\sqrt{gQH}\rho C_P T}{2\sqrt{2}\sigma T_{\rm eff}^4}$$

- must find ∇!

computing ∇

- add and subtract $(\nabla_E - \nabla_{\mathrm{ad}}) \rightarrow$

$$A\,(\nabla - \nabla_E)^{3/2} + (\nabla - \nabla_E) + (\nabla_E - \nabla_{\mathrm{ad}}) = \nabla_R - \nabla_{\mathrm{ad}}$$

- $(\nabla_E - \nabla_{\mathrm{ad}}) = B\,(\nabla - \nabla_E)^{1/2}$
- $\rightarrow$ cubic equation for $x = (\nabla - \nabla_E)^{1/2}$

$$Ax^3 + x^2 + Bx = \nabla_R - \nabla_{\mathrm{ad}}$$

- can be solved by standard analytic methods

computing ∇

- if x is the solution $\rightarrow$

$$(\nabla_R - \nabla) + x^2 + Bx = \nabla_R - \nabla_{\mathrm{ad}}$$

  so that

$$\nabla = \nabla_{\mathrm{ad}} + Bx + x^2$$

- with definition of $B \rightarrow$

$$\nabla_E = \nabla - x^2$$

- $\rightarrow$ we have the true T-gradient!

MLT

- true T-gradient known $\rightarrow$
- integrate it together with hydrostatic
- $\rightarrow$ gives T-P structure of convective atmosphere
- outer atmosphere $\rightarrow$
- ρ and κ_{Ross} small
- $\rightarrow$ stable against convection
- very effective convection (VLMS): $\nabla \rightarrow \nabla_{\mathrm{ad}}$
- in this case, MLT not needed, pure adiabatic structure

MLT

- convection up to optically thin layers $\rightarrow$
- cannot use diffusion approximation to compute F_{rad}
- use F_{rad} from RTE solution
- iterate for correct T-structure!