Stellar/Planetary Atmospheres

Part 07: micro-physics of lines

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Topics

- micro-physics of line radiation
  - Einstein coefficients
  - 2 level atom rate equations
  - line transfer examples
  - 2 level atom with continuum
  - full rate equations

Einstein coefficients

- radiative transitions between 2 bound levels
- A_{21}: number of spontaneous radiative transitions from 2 (upper level) to 1 per unit time
- $B_{12}\bar{J}$: number of transitions from 1 to 2 due to absorption of photon per atom in level 1 per unit time
- $\bar{J} = \int \Phi(\nu) J_\nu \, d\nu$
- $B_{21}\bar{J}$: number of 2 to 1 radiative transitions stimulated by photons.
- A_{21}: Einstein coefficient for spontaneous emission
- B_{21}: Einstein coefficient for stimulated emission
- B_{12}: Einstein coefficient for absorption

Einstein coefficients

- n_1, n_2: number densities cm^{-3} of atoms in levels 1 and 2
- $n_2(A_{21} + B_{21}\bar{J})$: number of downward radiative transitions (cm$^{-3}s^{-1}$)
- $n_1 B_{12}\bar{J}$: number of upward radiative transitions (cm$^{-3}s^{-1}$)
- statistical equilibrium $\rightarrow$

$$n_2(A_{21} + B_{21}\bar{J}) = n_1 B_{12}\bar{J}$$

Einstein coefficients

- TE $\rightarrow \bar{J} = B_\nu(\nu_0)$ where ν_0 is the central frequency of the line
- TE $\rightarrow$
$$\frac{n_2}{n_1} = \frac{n_2^*}{n_1^*} = \frac{g_2}{g_1} \exp\left(-\frac{h\nu_0}{kT}\right)$$
- so that in TE

$$n_2^*(A_{21} + B_{21} B_\nu(\nu_0)) = n_1^* B_{12} B_\nu(\nu_0)$$

Einstein coefficients

- solving for $B_\nu(\nu_0) \rightarrow$

$$\begin{aligned} B_\nu(\nu_0) &= \frac{A_{21}}{\frac{n_1^*}{n_2^*} B_{12} - B_{21}} \\ &= \frac{A_{21}/B_{21}}{\exp\left(-\frac{h\nu_0}{kT}\right)\left(\frac{g_1 B_{12}}{g_2 B_{21}}\right) - 1} \end{aligned}$$

- Einstein coefficients are constants $\rightarrow$

$$\begin{aligned} \frac{A_{21}}{B_{21}} &= \frac{2h\nu_0^3}{c^2} \\ g_1 B_{12} &= g_2 B_{21} \end{aligned}$$

Collisional transitions

- $n_2 C_{21}$: number of downward collisional transitions ($\mathrm{cm}^{-3} s^{-1}$)
- $n_1 C_{12}$: number of upward collisional transitions ($\mathrm{cm}^{-3} s^{-1}$)
- C's depend on collision partner's number density, e.g., electron density
- statistical equilibrium equations with both radiative and collisional rates:

$$n_2 \left(A_{21} + B_{21} \bar{J} + C_{21} \right) = n_1 \left(B_{12} \bar{J} + C_{12} \right)$$

Collisional transitions

- C's do not depend on n_1, n_2, $\bar{J}$
- $\rightarrow$ in TE

$$n_2^* C_{21} = n_1^* C_{12}$$

statistical equations: limits

- let $\alpha = \frac{2h\nu_0^3}{c^2}$
- with this the stat. equations are

$$n_2 \left(1 + \frac{\bar{J}}{\alpha} + \frac{C_{21}}{A_{21}}\right) = n_1 \frac{g_2}{g_1} \left(\frac{\bar{J}}{\alpha} + \frac{C_{21}}{A_{21}} \exp\left(-\frac{h\nu_0}{kT}\right)\right)$$

- therefore

$$\frac{n_2}{n_1}\frac{g_1}{g_2} = \frac{\frac{\bar{J}}{\alpha} + \frac{C_{21}}{A_{21}} \exp\left(-\frac{h\nu_0}{kT}\right)}{1 + \frac{\bar{J}}{\alpha} + \frac{C_{21}}{A_{21}}}$$

statistical equations: limits

- 2 important limits:
  1. $C_{21} \gg A_{21} \rightarrow$
  $$\frac{n_2}{n_1} \rightarrow \frac{n_2^*}{n_1^*}$$
  2. $\bar{J} = B_\nu(\nu_0) \rightarrow$
  $$\frac{n_2}{n_1} \rightarrow \frac{n_2^*}{n_1^*}$$
  regardless of C_{21}

abs/em coefficients

- spontaneous emission:

$$n_2 A_{21} \Phi(\nu)$$

  number of spontaneous emissions ($\mathrm{cm}^{-3} s^{-1} \mathrm{Hz}^{-1}$)
- each emission produces energy $\approx h\nu_0$
- photons distributed over solid angle 4π
- $\rightarrow$ emission coefficient

$$j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \Phi(\nu)$$

abs/em coefficients

- absorption (without stimulated emission):
- total energy absorbed by the line ($\mathrm{cm}^{-3}s^{-1}$)

$$h\nu_0 n_1 B_{12} \bar{J}$$

- frequency dependent absorption ($\mathrm{cm}^{-3}s^{-1}\mathrm{Hz}^{-1}$):

$$\frac{h\nu}{4\pi} n_1 B_{12} \Phi(\nu) I_\nu$$

- $\rightarrow$ absorption coefficient without stimulated emission:

$$\kappa_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \Phi(\nu)$$

abs/em coefficients

- stimulated emission:
- frequency dependent emission ($\mathrm{cm}^{-3} s^{-1} \mathrm{Hz}^{-1}$):

$$\frac{h\nu}{4\pi} n_2 B_{21} \Phi(\nu) I_\nu$$

transfer equation

- pp RTE:

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

- inserting $\rightarrow$

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi} n_1 B_{12} \Phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21} \Phi(\nu) + \frac{h\nu}{4\pi} n_2 B_{21} \Phi(\nu) I_\nu$$

or

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi} \Phi(\nu) \left[\left(n_1 B_{12} - n_2 B_{21} \right) I_\nu - n_2 A_{21} \right]$$

abs. coeff w/ stim. emission

- rewrite κ_ν as

$$\begin{aligned} \kappa_\nu &= \frac{h\nu}{4\pi}\left(n_1 B_{12} - n_2 B_{21}\right)\Phi(\nu) \\ &= \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)\Phi(\nu) \end{aligned}$$

source function

- $S_\nu = j_\nu / \chi_\nu$
- insert emission and absorption coefficients $\rightarrow$

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$$

- thus

$$S_\nu = \frac{2h\nu^3/c^2}{\frac{n_1}{n_2}\frac{g_2}{g_1} - 1}$$

- $S_\nu = B_\nu$ if

$$\frac{n_2}{n_1} = \frac{n_2^*}{n_1^*}$$

source function

- if this is the case, then

$$\left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) = \left(1 - \exp\left(-\frac{h\nu}{kT}\right)\right)$$

- $S_\nu = B_\nu$ is condition for LTE
- κ_ν negative for 'population inversions':

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}$$

line source function

- statistical equilibrium equation without collisions

$$n_2 \left(A_{21} + B_{21}\bar{J}\right) = n_1 B_{12} \bar{J}$$

- with $\alpha = 2h\nu^3/c^2 \rightarrow$

$$n_2 \left(1 + \frac{\bar{J}}{\alpha}\right) = n_1 \frac{g_1}{g_2} \frac{\bar{J}}{\alpha}$$

so that

$$\frac{n_2}{n_1} \frac{g_1}{g_2} = \frac{\bar{J}/\alpha}{1 + \bar{J}/\alpha}$$

line source function

- source function $\rightarrow$

$$S = \frac{\alpha}{\frac{n_1}{n_2}\frac{g_2}{g_1} - 1} = \frac{\alpha}{\frac{1+\bar{J}/\alpha}{\bar{J}/\alpha} - 1}$$

- or $S_\nu = \bar{J}$
- for the collision-free case!

general line source function

- write pp RTE for the line as

$$\frac{dI_\nu}{ds} = \kappa_\nu \left(I_\nu - S_\nu \right)$$

- with

$$\kappa_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left(1 - \frac{n_2}{n_1} \frac{g_1}{g_2} \right) \Phi(\nu)$$

  and

$$S_\nu = \frac{2h\nu^3/c^2}{\frac{n_1}{n_2}\frac{g_2}{g_1} - 1}$$

general line source function

- write statistical eq. equation →

$$n_2 \left(A_{21} + B_{21}\bar{J} + C_{21}\right) = n_1 \left(B_{12}\bar{J} + C_{12}\right)$$

- rewrite with the line source function S →

$$S = \frac{\bar{J} + \tilde{\epsilon}B}{1 + \tilde{\epsilon}}$$

with

$$\tilde{\epsilon} = \frac{C_{21}}{A_{21}}\left(1 - \exp\left(-\frac{h\nu}{kT}\right)\right)$$

general line source function

- this can be re-cast by defining

$$\epsilon = \frac{\tilde{\epsilon}}{1+\tilde{\epsilon}}$$

into

$$S = (1-\epsilon)\bar{J} + \epsilon B$$

- this is the same form as for the continuum source function with scattering that we defined earlier but with $J_\nu \to \bar{J}$!

analytic solutions

- assume Gauss line profiles:

$$\Phi(x) = \frac{1}{\Delta\nu_D\sqrt{\pi}} \exp(-x^2)$$

with

$$x = \frac{\nu - \nu_0}{\Delta\nu_D}$$

- assume $\Delta\nu_D = \mathrm{const.}$ for simplicity
- let

$$d\tau = dr \frac{1}{\Delta\nu_D} \frac{h\nu}{4\pi} n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)$$

analytic solutions

- $\rightarrow$ form of the transfer equation

$$\mu \frac{dI_x}{d\tau} = \Phi_x (I_x - S)$$

with

$$\begin{aligned} \Phi_x &= \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right) \\ S &= (1-\epsilon)\bar{J} + \epsilon B \\ \bar{J} &= 2 \int \Phi_x J_x \, dx \end{aligned}$$

- τ is $\sqrt{\pi}$ time the line center optical depth!
- note frequency coupling due to $\bar{J}$.

analytic solutions

- formal solution $\rightarrow$

$$J_x(\tau) = \frac{1}{2} \int \Phi_x E_1(\Phi_x |t - \tau|) S(t) \, dt$$

- define a kernel function

$$K(p) = \int \Phi_x^2 E_1(\Phi_x p) \, dx$$

- normalization

$$\int K(p) \, dp = \frac{1}{2}$$

analytic solutions

- with this:

$$S(\tau) = (1-\epsilon)\int K(|t-\tau|)S(t)\,dt + \epsilon B$$

- $K(p)$ is complicated
- simplifications (e.g., approximate E_1 or simplify Φ_x) do not help much either
- $\rightarrow$ numerical solutions for $B = 1 \rightarrow$
- from Averett, 'Solutions of the two-level line transfer Problem with complete redistribution' (1965)

solutions for $B = 1$

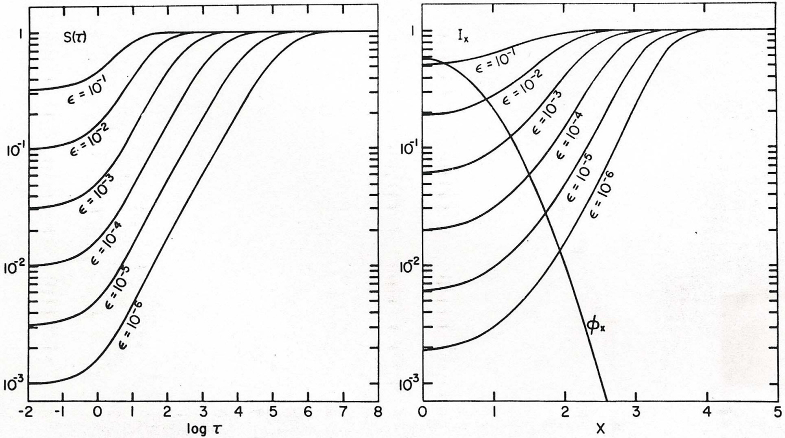

Figure 1.--$S(\tau)$ and I_x for $B = 1$, $\phi_x = \frac{1}{\sqrt{\pi}} e^{-x^2}$, $r = 0$, and constant values of ϵ.

- note thermalization depth $\propto 1/\epsilon$

solutions for $B = 1 + 100\,\mathbf{exp}(-c\tau)$

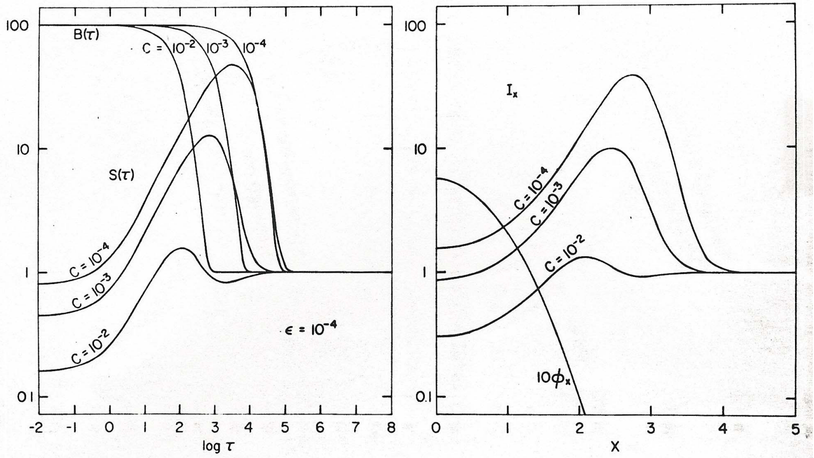

Figure 11.--Solutions with $B(\tau) = 1 + 100e^{-ct}$ and $\phi_x = (1/\sqrt{\pi})\, e^{-x^2}$.

solutions for $B = 1 + 100\,\exp(-c\tau)$

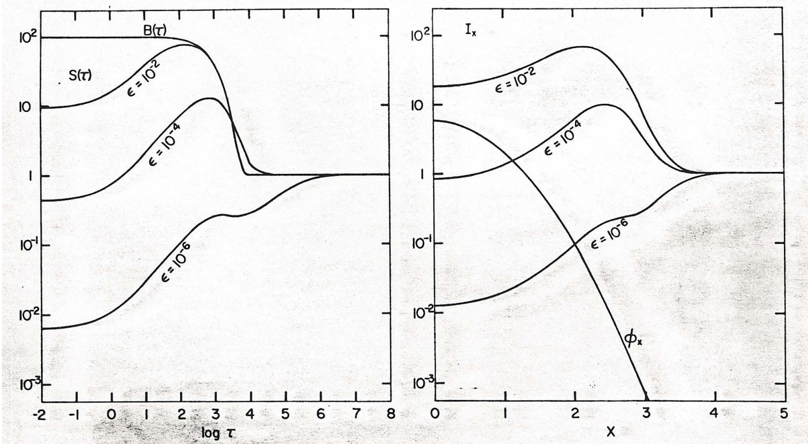

Figure 13.--Solutions with $B(\tau) = 1 + 100e^{-.001\tau}$, $\phi_x = (1/\sqrt{\pi})\, e^{-x^2}$, and $\epsilon = 10^{-2}$, 10^{-4}, 10^{-6}.

line wings

- $x \to \infty$: $\Phi_x \to 0$
- other processes will become important
- example: continuum opacities (background)
- write RTE as

$$\mu \frac{dI_x}{d\tau} = \Phi_x (I_x - S) + r(I_x - S_c)$$

- r: ratio continuum to line-center opacity (sans $\sqrt{\pi}$)
- S_c: continuum source function, e.g., $S_c = B$

solutions with background

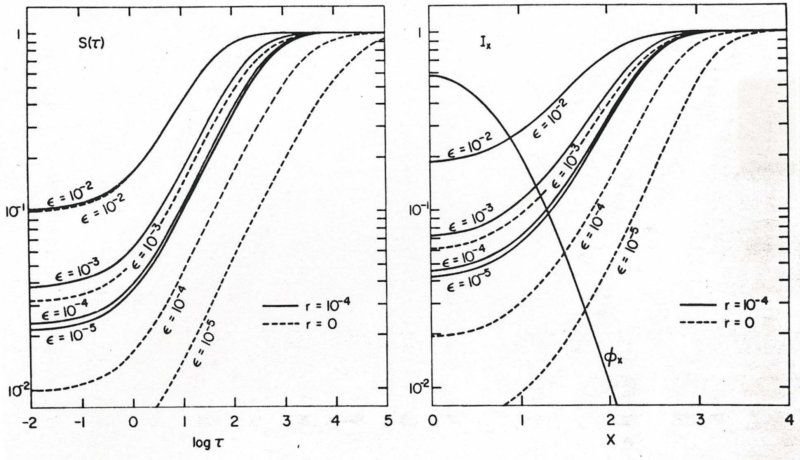

Figure 8.--Solutions with constant values of r and ε. r is unimportant when r ≪ ε; ε is unimportant when ε ≪ r.

line wings

- solution independent of ϵ if $\epsilon \ll r$
- for $\epsilon \gg r$ but still $\epsilon \ll 1$
  - compare to A_{21} radiative emission followed by $B_{12}\bar{J}$ absorption $\rightarrow$ scattering
  - $\rightarrow$ no coupling of radiation and temperature
  - small probability of collisional de-excitation
  - $\rightarrow$ causes $S \rightarrow B$

line wings

- for $r \gg \epsilon$ but $r \ll 1$
  - continuum emission and absorption take place of collisions
  - $\rightarrow S \rightarrow S_c$
  - $\rightarrow$ thermalization depth no longer $1/\epsilon$
  - will be between $1/\sqrt{r}$ and $1/r$

solutions with Voigt profiles

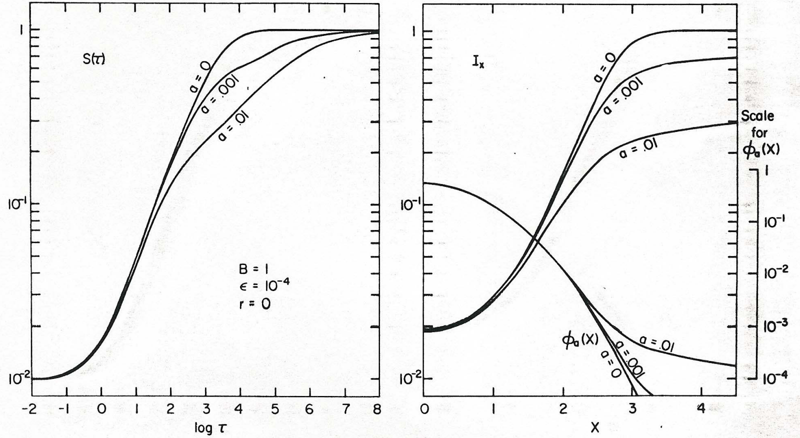

Figure 4.--Solutions with Doppler and Voigt profiles.

solutions with Voigt profiles

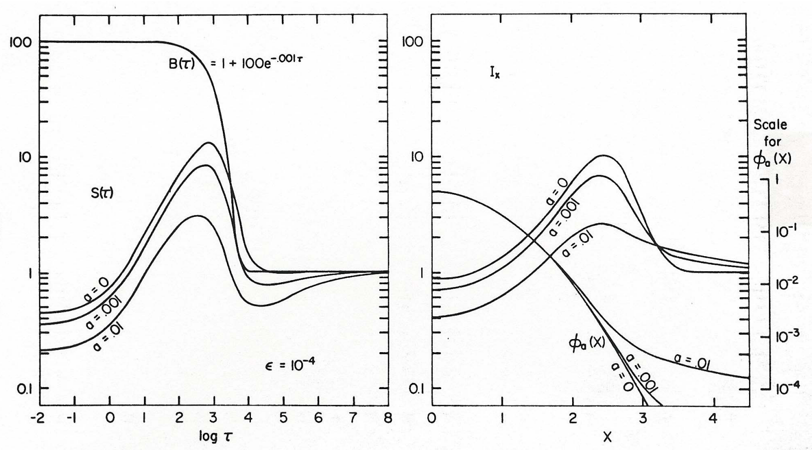

Figure 15.--Solutions with Doppler and Voigt profiles and variable B.

solutions with Voigt profiles

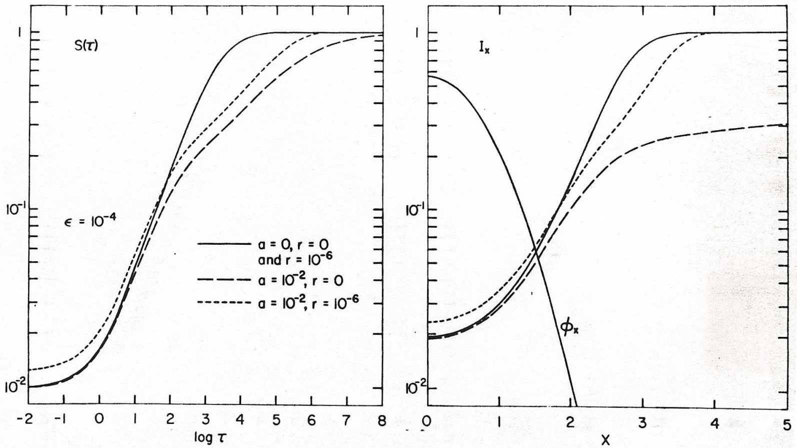

Figure 9.--Solutions with Doppler and Voigt profiles and with r = 0 and 10^{-6}.

solutions with $\epsilon = \epsilon(\tau)$

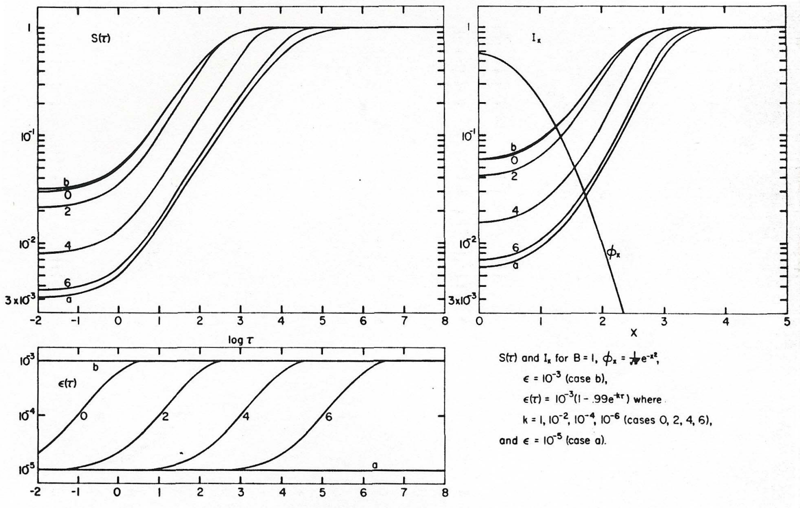

Figure 3.--Solutions for $10^{-5} \leq \epsilon(\tau) \leq 10^{-3}$.

2-level atom with continuum

- allow transitions from/to continuum
- $P_{2\kappa}$: number of transitions from level 2 to continuum
- $P_{\kappa 2}$: number of transitions to level 2 from continuum
- with this the stat. equation for level 2 is

$$n_2 \left(A_{21} + B_{21} \bar{J} + C_{21} + P_{2\kappa} \right) = n_1 \left(B_{12} \bar{J} + C_{12} \right) + n_\kappa P_{\kappa 2}$$

- statistical equilibrium equation for the continuum:

$$n_\kappa (P_{\kappa 1} + P_{\kappa 2}) = n_1 P_{1\kappa} + n_2 P_{2\kappa}$$

2-level atom with continuum

- adding the last 2 equations $\rightarrow$

$$n_1 \left(B_{12}\bar{J} + C_{12} + P_{1\kappa} \right) = n_2 \left(A_{21} + B_{21}\bar{J} + C_{21} \right) + n_\kappa P_{1\kappa}$$

- $\rightarrow$ statistical equilibrium equation for level 1

2-level atom with continuum

- use continuum equation to eliminate n_κ from level 2 equation $\rightarrow$

$$
\begin{aligned}
n_2\left(A_{21}+B_{21}\bar{J}+C_{21}+P_{2\kappa}\right) &= n_1\left(B_{12}\bar{J}+C_{12}\right) \\
&\quad +\frac{n_1 P_{1\kappa}P_{\kappa 2}+n_2 P_{2\kappa}P_{\kappa 2}}{P_{\kappa 1}+P_{\kappa 2}}
\end{aligned}
$$

- define

$$
\begin{aligned}
P_{12} &= \frac{P_{1\kappa}P_{\kappa 2}}{P_{\kappa 1}+P_{\kappa 2}} \\
P_{21} &= \frac{P_{2\kappa}P_{\kappa 1}}{P_{\kappa 1}+P_{\kappa 2}}
\end{aligned}
$$

2-level atom with continuum

- with this

$$n_2 \left(A_{21} + B_{21} \bar{J} + C_{21} + P_{21} \right) = n_1 \left(B_{12} \bar{J} + C_{12} + P_{12} \right)$$

continuum rates

- write

$$P = R + C$$

  with

- $R_{l\kappa}$: photoionization rate
- $R_{\kappa l}$: radiative recombination rate
- $C_{l\kappa}$: collisional ionization rate
- $C_{\kappa l}$: collisional recombination rate
- all per initial-state atom or ion

continuum rates

- $\alpha_l(\nu)$: b-f absorption coefficient from level l
- ν_l: threshold frequency for level l
- photoionization rate $\rightarrow$

$$R_{l\kappa} = \int d\Omega \int_{\nu_l}^{\infty} d\nu \frac{1}{h\nu} \alpha_l(\nu) I_\nu$$

or

$$R_{l\kappa} = 4\pi \int_{\nu_l}^{\infty} J_\nu \frac{1}{h\nu} \alpha_l(\nu)\, d\nu$$

continuum rates

- $\alpha_l^r(\nu)$: spontaneous radiative recombination coefficient to level l
- $\alpha_l^{sr}(\nu)$: stimulated radiative recombination coefficient to level l
- $\rightarrow$ transfer equation

$$\frac{dI_\nu}{ds} = -n_l \alpha_l(\nu) I_\nu + n_\kappa \alpha_l^{sr}(\nu) I_\nu + n_\kappa \alpha_l^r(\nu)$$

continuum rates

- $\int d\Omega \int_{\nu_l}^{\infty} d\nu \frac{1}{h\nu} \rightarrow$

$$\int d\Omega \int_{\nu_l}^{\infty} d\nu \frac{1}{h\nu}\frac{dI_\nu}{ds} =$$
$$-n_l 4\pi \int_{\nu_l}^{\infty} \frac{1}{h\nu}\alpha_l(\nu) J_\nu \, d\nu$$
$$+n_\kappa 4\pi \int_{\nu_l}^{\infty} \frac{1}{h\nu}\left[\alpha_l^{sr}(\nu) J_\nu + \alpha_l^{r}(\nu)\right] \, d\nu$$

continuum rates

- in TE:
  - $dI_\nu/ds = 0$
  - $J_\nu = B_\nu$
  - $n_l/n_\kappa = (n_l/n_\kappa)^*$
- therefore

$$\left(\frac{n_l}{n_\kappa}\right)^* \int_{\nu_l}^{\infty} \frac{1}{h\nu} \alpha_l(\nu) B_\nu \, d\nu = \int_{\nu_l}^{\infty} \frac{1}{h\nu} \left[\alpha_l^{sr}(\nu) B_\nu + \alpha_l^{r}(\nu)\right] \, d\nu$$

- and thus

$$\left(\frac{n_l}{n_\kappa}\right)^* \alpha_l(\nu) B_\nu = \alpha_l^{sr}(\nu) B_\nu + \alpha_l^{r}(\nu)$$

continuum rates

- this can be written as

$$\frac{2h\nu^3/c^2}{\exp\left(\frac{h\nu}{kT}\right)-1} = \frac{\alpha_l^r/\alpha_l^{sr}}{\left(\frac{n_l}{n_\kappa}\right)^* \frac{\alpha_l}{\alpha_l^{sr}} - 1}$$

  for all T

- in analogy with the relations between the Einstein coefficients we write

$$\begin{aligned} \alpha_l^r &= \frac{2h\nu^3}{c^2}\alpha_l^{sr} \\ \alpha_l^{sr} &= \alpha_l \left(\frac{n_l}{n_\kappa}\right)^* \exp\left(-\frac{h\nu}{kT}\right) \end{aligned}$$

continuum rates

- the radiative recombination rate:

$$R_{\kappa l} = 4\pi \int_{\nu_l}^{\infty} \frac{1}{h\nu} \left[\alpha_l^{sr}(\nu) J_\nu + \alpha_l^{r}(\nu)\right] d\nu$$

  or

$$R_{\kappa l} = \left(\frac{n_l}{n_\kappa}\right)^{*} 4\pi \int_{\nu_l}^{\infty} \frac{1}{h\nu} \alpha_l(\nu) \exp\left(-\frac{h\nu}{kT}\right) \left(\frac{2h\nu^3}{c^2} + J_\nu\right) d\nu$$

- If $J_\nu \ll 2h\nu^3/c^2 \rightarrow$ rate depends only on T and n_e (via n_κ) for each l
- can be tabulated, e.g., for H I in gaseous nebulae

continuum rates

- collisional ionization and recombination rates
- for b-b rates:

$$n_1^* C_{12} = n_2^* C_{21}$$

- in analogy

$$C_{\kappa l} = \left(\frac{n_l}{n_\kappa}\right)^* C_{l\kappa}$$

- similar for radiative continuum rates!

statistical equilibrium

- statistical equilibrium equation:

$$n_2\left(A_{21}+B_{21}\bar{J}+C_{21}+P_{21}\right)=n_1\left(B_{12}\bar{J}+C_{12}+P_{12}\right)$$

- use definition of P_{12} to define

$$P_{12}=\frac{P_{1\kappa}P_{\kappa 2}}{P_{\kappa 1}+P_{\kappa 2}}\equiv P'_{21}\left(\frac{n_2^*}{n_1^*}\right)$$

- this gives

$$n_2\left(1+\frac{\bar{J}}{2h\nu^3/c^2}+\frac{C_{21}+P_{21}}{A_{21}}\right)=$$
$$n_1\frac{g_2}{g_1}\left(\frac{\bar{J}}{2h\nu^3/c^2}+\frac{C_{21}+P'_{21}}{A_{21}}\exp\left(-\frac{h\nu}{kT}\right)\right)$$

source function

- insert the result into the expression for S:

$$
\begin{aligned}
S &= \frac{2h\nu^3/c^2}{\frac{n_1}{n_2}\frac{g_2}{g_1} - 1} \\
&= \frac{2h\nu^3/c^2}{\frac{1+\frac{\bar{J}}{2h\nu^3/c^2}+\frac{C_{21}+P_{21}}{A_{21}}}{\frac{\bar{J}}{2h\nu^3/c^2}+\frac{C_{21}+P'_{21}}{A_{21}\exp\left(-\frac{h\nu}{kT}\right)}-1}} \\
&= \frac{\bar{J} + \frac{C_{21}+P'_{21}}{A_{21}}\frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right)}{1 + \frac{1}{A_{21}}\left[C_{21} + P_{21} - (C_{21} + P'_{21})\exp\left(-\frac{h\nu}{kT}\right)\right]}
\end{aligned}
$$

source function

- this can be re-cast into

$$S = \frac{\bar{J} + \tilde{\epsilon}\tilde{B}}{1 + \tilde{\epsilon}}$$

with

$$\tilde{\epsilon} = \frac{C_{21} + P_{21}}{A_{21}} \left(1 - \frac{C_{21} + P'_{21}}{C_{21} + P_{21}} \exp\left(-\frac{h\nu}{kT}\right)\right)$$

and

$$\tilde{B} = \frac{2h\nu^3/c^2}{\frac{C_{21}+P_{21}}{C_{21}+P'_{21}} \exp\left(\frac{h\nu}{kT}\right) - 1}$$

- if $P_{21} \ll C_{21}$ and $P'_{21} \ll C_{21}$ this reduces to the previous case!

collision free case

- if collisions can be neglected $\rightarrow$

$$\tilde{\epsilon} = \frac{R_{2\kappa}}{A_{21}} \frac{1}{1 + \frac{n_2^*}{n_1^*} \frac{R_{2\kappa}^+}{R_{1\kappa}^+}} \left(1 - \frac{R_{1\kappa} R_{2\kappa}^+}{R_{2\kappa} R_{1\kappa}^+} \exp\left(-\frac{h\nu}{kT}\right)\right)$$

and

$$\tilde{B} = \frac{2h\nu^3/c^2}{\frac{R_{2\kappa} R_{1\kappa}^+}{R_{1\kappa} R_{2\kappa}^+} \exp\left(\frac{h\nu}{kT}\right) - 1}$$

with

$$R_{\kappa l} = \left(\frac{n_l}{n_\kappa}\right)^* R_{l\kappa}^+$$

collision free case

- hydrogenic cross-section $\rightarrow$

$$\alpha_I(\nu) = \alpha_I \left(\frac{\nu_I}{\nu}\right)^3$$

- let $J_\nu \ll 2h\nu^3/c^2$ and set

$$J_\nu = 2h\nu^3/c^2 \exp\left(-\frac{h\nu}{kT_r}\right)$$

collision free case

- continuum rates $\rightarrow$

$$
\begin{aligned}
R^{+}_{l\kappa} &= 4\pi \int_{\nu_l}^{\infty} \frac{\alpha_l}{h\nu} \left(\frac{\nu_l}{\nu}\right)^3 \exp\left(-\frac{h\nu}{kT}\right)\, d\nu \\
&= \frac{8\pi\alpha_l\nu_l^3}{c^2} \int_{\nu_l}^{\infty} \frac{1}{\nu} \exp\left(-\frac{h\nu}{kT}\right)\, d\nu \\
&= \frac{8\pi\alpha_l\nu_l^3}{c^2} E_1\left(\frac{h\nu_l}{kT}\right)
\end{aligned}
$$

and

$$
R_{l\kappa} = \frac{8\pi\alpha_l\nu_l^3}{c^2} E_1\left(\frac{h\nu_l}{kT_r}\right)
$$

collision free case

- this gives

$$\tilde{\epsilon} \approx \frac{R_{2\kappa}}{A_{21}}$$

- in the regime where

$$E_1(x) \approx \frac{\exp(-x)}{x}$$

we get

$$\tilde{B} \approx 2h\nu^3/c^2 \exp\left(-\frac{h\nu}{kT_r}\right)$$

collision free case

- photoionization dominated case
- $\tilde{B}$ only depends on radiation temperature T_r
- not on the local kinetic temperature T!