Stellar/Planetary Atmospheres

Part 06: line transfer

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Topics

- Classical treatment of line transfer
  - Milne-Eddington Model
  - scattering lines
  - absorptive lines
  - Schuster mechanism
  - curve of growth

Classical line transfer

- assumes 2 types of lines:
  - scattering lines
  - absorptive lines
- assume that fraction $(1 - \epsilon)$ of photons interacting with line are scattered
- $\rightarrow$ no contact to thermal pool
- assume that scattering is coherent & isotropic
- fraction ϵ of line photons thermalize

Classical line transfer

- $\rightarrow$ line emission coefficient

$$\eta_l = \Phi_\nu \chi_l \left(\epsilon B_\nu + (1-\epsilon) J_\nu\right)$$

- Φ_ν: normalized line profile
- χ_l: total line extinction coefficient
- J_ν: mean intensity

Classical line transfer

- pp RTE in the line:

$$\mu \frac{dI_\nu}{dz} = -\chi_\nu I_\nu + \kappa_c B_\nu + \sigma_c J_\nu + \Phi_\nu \chi_l \left(\epsilon B_\nu + (1-\epsilon) J_\nu\right)$$

- and

$$\chi_\nu = \kappa_c + \sigma_c + \Phi_\nu \chi_l$$

- with the optical depth

$$d\tau = -\chi_\nu dz$$

Classical line transfer

- define

$$\begin{aligned} \rho &= \frac{\sigma_c}{\kappa_c + \sigma_c} \\ \beta_\nu &= \frac{\chi_l \Phi_\nu}{\kappa_c + \sigma_c} \end{aligned}$$

- to obtain

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \frac{[(1-\rho) + \epsilon\beta_\nu]\, B_\nu + [\rho + (1-\epsilon)\beta_\nu]\, J_\nu}{1 + \beta_\nu}$$

- define

$$\lambda_\nu = \frac{[(1-\rho) + \epsilon\beta_\nu]}{1 + \beta_\nu}$$

Classical line transfer

- $\rightarrow$ simplify to

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - (1 - \lambda_\nu) J_\nu - \lambda_\nu B_\nu$$

- *Milne-Eddington equation*

Milne-Eddington equation

- assume λ_ν, ϵ, ρ are independent of τ
- assume $\kappa_c = \mathrm{const.}$ over the line
- to solve Milne-Eddington equation assume linear B_ν:

$$B_\nu = a + b\tau_\nu = a + \frac{b\tau_\nu}{1+\beta_\nu} = a + p_\nu \tau_\nu$$

- take the 0th moment of the M-E equation

$$\frac{dH_\nu}{d\tau_\nu} = J_\nu - (1-\lambda_\nu)J_\nu - \lambda_\nu B_\nu = \lambda_\nu (J_\nu - B_\nu)$$

Milne-Eddington equation

- in the Eddington approximation $K_\nu/J_\nu = 1/3$ the solution is (see before!)

$$J_\nu = a + p_\nu\tau_\nu + \left(p_\nu - \sqrt{3}a\right)\exp\left(-\sqrt{3\lambda_\nu}\tau_\nu\right)\frac{1}{\sqrt{3}+\sqrt{3\lambda_\nu}}$$

- the emergent flux is

$$H_\nu(0) = \frac{J_\nu(0)}{\sqrt{3}} = \frac{1}{3}\frac{p_\nu + \sqrt{3\lambda_\nu}a}{1+\sqrt{\lambda_\nu}}$$

Milne-Eddington equation

- thermalization depths (where $J_\nu \to B_\nu$):

$$\tau_{\rm th} = \frac{1}{\sqrt{\lambda_\nu}}$$

- continuum: $\beta_\nu = 0 \to$

$$\tau_{\rm th} = (1-\rho)^{-1/2}$$

- very strong line: $\beta_\nu \to \infty$

$$\tau_{\rm th} = \frac{1}{\sqrt{\epsilon}}$$

for isotropic & coherent scattering

Milne-Eddington equation

- detailed analysis for the more realistic case of
- *complete redistribution* (CRD) where

$$\eta_I = \Phi_\nu \chi_I \left(\epsilon B_\nu + (1-\epsilon)\bar{J} \right)$$

which $\bar{J} = \int \Phi_\nu J_\nu \, d\nu$

Milne-Eddington equation

- gives these results for different line profiles:
  - Gauss profile:

$$\tau_{\mathrm{th}} = \frac{1}{\sqrt{\epsilon}}$$

  - Lorenz profile:

$$\tau_{\mathrm{th}} = \frac{1}{\sqrt{\epsilon^2}}$$

  - Voigt profile:

$$\tau_{\mathrm{th}} = \frac{\alpha}{\sqrt{\epsilon^2}}$$

emergent line profile

- use $H_\nu(0)$ to compute flux profile of the line
- continuum: $(\beta_\nu = 0)$, $\lambda_\nu = (1 - \rho)$ so that

$$H_c(0) = \frac{1}{3}\frac{b + a\sqrt{3(1-\rho)}}{1 + \sqrt{1-\rho}}$$

- 'residual flux' $R_\nu = H_\nu/H_c$ in the line:

$$R_\nu = \left[\frac{p_\nu + \sqrt{3\lambda_\nu}a}{1 + \sqrt{\lambda_\nu}}\right]\left[\frac{1 + \sqrt{1-\rho}}{b + a\sqrt{3(1-\rho)}}\right]$$

- 2 limiting cases:

scattering lines

- set $\rho = 0$ (no continuum scattering)
- set $\epsilon = 0$ (pure scattering in the line)
- $\rightarrow \lambda_\nu = 1/(1+\beta_\nu)$ and

$$R_\nu = 2\frac{\left[\frac{b}{1+\beta_\nu} + a\sqrt{\frac{3}{1+\beta_\nu}}\right]}{\left(1+\sqrt{\frac{1}{1+\beta_\nu}}\right)\left(\sqrt{3}a+b\right)}$$

- $\beta_\nu \rightarrow \infty$ (very strong line) $\rightarrow$

$$R_\nu = 0$$

- $\rightarrow$ very strong scattering line can be totally 'black'

absorptive lines

- set $\rho = 0$ (no continuum scattering)
- set $\epsilon = 1$ (pure absorption in the line)
- $\rightarrow \lambda_\nu = 1$ and

$$R_\nu = \frac{\sqrt{3}a + b/(1+\beta_\nu)}{\sqrt{3}a + b}$$

- $\beta_\nu \rightarrow \infty$ (very strong line) $\rightarrow$

$$R_0 = \frac{1}{1 + b/\sqrt{3}a}$$

absorptive lines

- in terms of B_ν and its gradient

$$B_\nu(\bar{\tau}) = B_\nu(T_0) + \frac{dB_\nu}{d\bar{\tau}}\bar{\tau} = B_0 + B_1\bar{\tau}$$

- set (from grey $T(\tau)$)

$$T^4 = T_0^4(1 + \frac{3}{2}\bar{\tau})$$

absorptive lines

- this gives

$$R_0 = \left(1 + \left[\sqrt{3}X_0(\bar{\kappa}/\kappa)/8\right]^{-1}\right)^{-1}$$

with

$$X_0 = \frac{u_0}{1 - \exp(-u_o)}$$

and

$$u_0 = \frac{h\nu}{kT_0}$$

absorptive lines

- Sun: $T_0 \approx 4800\,\mathrm{K}$, $\lambda = 5000\,\text{Å}$, $u_0 \approx 6$
- so that $X_0 \approx 6$, $\bar{\kappa} \approx \kappa$
- this gives
$$R_0 \approx 0.44$$
  which is in fair agreement with many observed solar lines

Schuster mechanism

- consider continuum scattering
- $\rightarrow$ picture changes significantly
- assume $\rho = 1$ (scattering dominates continuum) $\rightarrow$

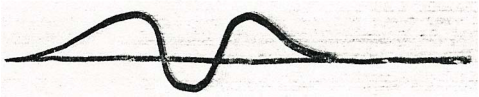
$$\lambda_\nu = \frac{\epsilon\beta_\nu}{1+\beta_\nu}$$

and

$$R_\nu = \frac{\left[\frac{1}{1+\beta_\nu} + \left(\frac{a}{b}\right)\sqrt{\frac{3\epsilon\beta_\nu}{1+\beta_\nu}}\right]}{\left[1+\sqrt{\frac{\epsilon\beta_\nu}{1+\beta_\nu}}\right]}$$

Schuster mechanism

- $\epsilon = 0 \rightarrow R_\nu = 1/(1 + \beta_\nu)$: line is in absorption
- $\epsilon = 1$, $\beta_\nu \rightarrow \infty \rightarrow R_\nu \rightarrow (\sqrt{3}/2)(a/b)$: line can be in absorption or emission, depending on a/b.
  - $a/b > 2/\sqrt{3} \rightarrow$ line in emission for all β_ν
  - $a/b = 2/\sqrt{3} \rightarrow$ line core and extreme wing on continuum, elsewhere in emission
  - $1/\sqrt{3} < a/b < 2/\sqrt{3} \rightarrow$ drawing
  - $a/b < 1/\sqrt{3} \rightarrow$ absorption feature



curve of growth

- define *equivalent width* of a line

$$W_\lambda = \int \left(1 - \frac{H_\lambda}{H_c}\right) d\lambda = \int (1 - R_\lambda)\, d\lambda$$

- curve of growth gives W_λ as function of the number of absorbing atoms

curve of growth

- simple approximation: assume line forms in homogeneous layer at given (T, P_e)
- LTE line absorption coefficient

$$\chi_{ij}(\nu) = \Phi_\nu \chi_{ij}$$

- line profile $\rightarrow$ Voigt function $H(a, v)$
- assume also $a = \mathrm{const.}$
- $\beta_\nu = \chi_{ij}(\nu)/\chi_c$
- independent of τ!
- absorptive line

curve of growth

- $\rightarrow$ emergent flux

$$F_\nu = 2 \int B_\nu(T(\tau)) E_2 \left(\int_0^\tau (1 + \beta_\nu) \, dt \right) (1 + \beta_\nu) \, d\tau$$

curve of growth

- linear Planck function

$$B_\nu(T(\tau)) = B_0 + B_1\tau$$

- $\rightarrow$

$$F_\nu = B_0 + \frac{2}{3}\left(\frac{B_1}{1+\beta_\nu}\right)$$

- continuum flux

$$F_c = B_0 + \frac{2}{3}B_1$$

curve of growth

- depth of the line

$$A_\nu = 1 - R_\nu = 1 - \frac{F_\nu}{F_c} = \frac{\frac{\beta_\nu}{1+\beta_\nu}}{1 + \frac{3}{2}\frac{B_0}{B_1}}$$

- define for $\beta_\nu \to \infty$

$$A_0 = \left[1 + \frac{3}{2}\frac{B_0}{B_1}\right]^{-1}$$

- so that

$$A_\nu = A_0 \frac{\beta_\nu}{1 + \beta_\nu}$$

curve of growth

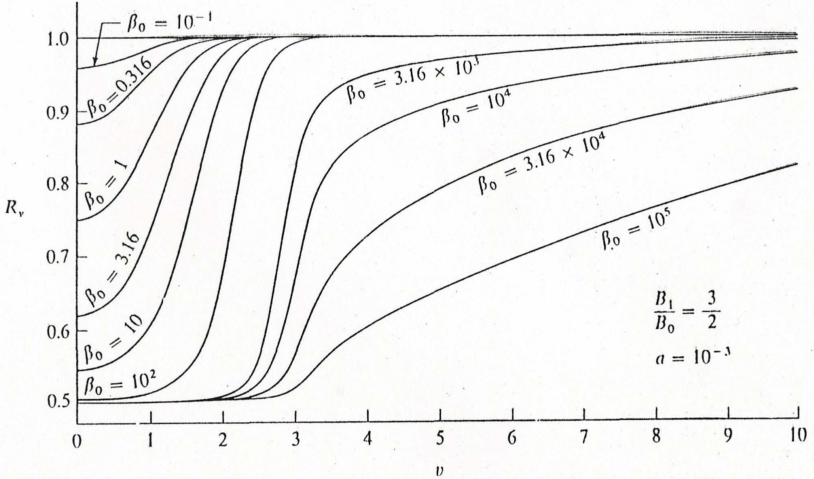


FIGURE 10-1

Development of a spectrum line with increasing number of atoms along the line of sight. The line is assumed to be formed in pure absorption. For $\beta_0 \lesssim 1$, the line strength is directly proportional to the number of absorbers. For $30 \lesssim \beta_0 \lesssim 10^3$ the line is saturated, but the wings have not yet begun to develop. For $\beta_0 \gtrsim 10^4$ the line wings are strong and contribute most of the equivalent width.

curve of growth

- compute $W_\lambda \rightarrow$

$$W_\lambda = \int_{-\infty}^{\infty} A_\nu \, d\Delta\nu = 2A_0 \Delta\nu_D \int_0^{\infty} \frac{\beta(\nu)}{1+\beta(\nu)} \, d\nu$$

- factor '2': line symmetric around center and

$$\beta(v) = \frac{\chi_{ij}}{\kappa_c} H(a, v) = \beta_0 H(a, v)$$

curve of growth

- convenient to define *reduced equivalent width*

$$W^* = \frac{W_\lambda}{2A_0\Delta\nu_D}$$

- so that

$$W^* = \int_0^\infty \frac{\beta_0 H(a,v)}{1+\beta_0 H(a,v)}\,dv = W^*(a,\beta_0)$$

curve of growth

- limiting cases:
- use schematic Voigt profile

$$H(a, v) = \exp(-v^2) + \frac{a}{\sqrt{\pi} v^2}$$

to analyze behavior of $W^*(a, \beta_0)$

curve of growth

- for $\beta_0 < 1$: contribution by line core

$$
\begin{aligned}
W^* &= \beta_0 \int_0^\infty \exp(-v^2) \left[1 + \beta_0 \exp(-v^2)\right]^{-1} \, dv \\
&= \beta_0 \int_0^\infty \exp(-v^2) \left(1 - \beta_0 \exp(-v^2) + \cdots\right) \, dv \\
&= \frac{1}{2}\sqrt{\pi}\beta_0 \left(1 - (\beta_0/\sqrt{2}) + (\beta_0^2/\sqrt{3}) - \cdots\right) \, dv
\end{aligned}
$$

- weak lines $\rightarrow$ W^* depends linearly on the number of absorbers, independent of $\Delta\nu_D$

curve of growth

- 'saturation' part:
- line core at maximal depths, line wings still weak
- approximate

$$W^* \approx \sqrt{\ln \beta_0}\{1 - [\pi^2/24(\ln \beta_0)^2] - \cdots\}$$

  (semi-convergent series)
- $\rightarrow W^* \propto \sqrt{\ln \beta_0}$, weak dependence!

curve of growth

- very large β_0:
- line wings dominate $W^* \rightarrow$

$$W^* = \int_0^\infty \left(1 + \frac{v^2}{c}\right)^{-1} dv = \frac{1}{2}\pi c$$

with $c = a\sqrt{\beta_0}/\sqrt{b}$

- $\rightarrow$ 'damping' or $\sqrt{}$ part of the curve of growth

curve of growth

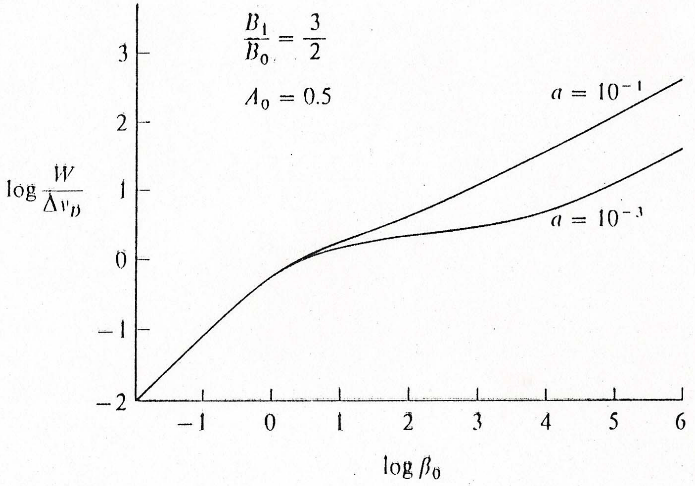


FIGURE 10-2
Curves of growth for pure absorption lines. Note that the larger the value of a, the sooner the square-root part of the curve rises away from the flat part.