Stellar/Planetary Atmospheres

Part 05: Numerical RT I

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Topics

- Numerical RT part 1
  - The operator splitting method
  - a method for the FS
  - computing Λ^*

Numerical solution

- there are *many* numerical RT methods
- too many to describe them all
  - direct methods
  - iterative methods
- all methods need to deal with scattering!
- here: I will describe the currently most used method

Numerical solution

- recap: scattering $\rightarrow$

$$S = (1-\epsilon)J + \epsilon B$$

$$\epsilon = \frac{\kappa}{\kappa + \sigma}$$

  - S depends on B and (unknown) J
  - $\rightarrow$ self-consistent solution for J required
  - direct solution expensive
  - use iterative method
  - eigenvalues of iteration matrix close to unity $\rightarrow$ use operator splitting to reduce eigenvalues of amplification matrix

Operator Splitting in RT

- Formal solution (how? see below!)

$$J = \Lambda S$$

- Λ-iteration:

$$\bar{J}_{\rm new} = \Lambda S_{\rm old}, \quad S_{\rm new} = (1-\epsilon)\bar{J}_{\rm new} + \epsilon B$$

- does *not* work for $\tau >> 1$ & small ϵ

Operator Splitting in RT

- Solution: Split Λ operator

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$$

- $\rightarrow$ iteration procedure

$$\bar{J}_{\mathrm{new}} = \Lambda^* S_{\mathrm{new}} + (\Lambda - \Lambda^*) S_{\mathrm{old}}$$

Operator Splitting in RT

- using

$$S = (1 - \epsilon)J + \epsilon B$$

  and

$$\bar{J}_{\mathrm{fs}} = \Lambda S_{\mathrm{old}}$$

- gives

$$\left[1 - \Lambda^*(1 - \epsilon)\right] \bar{J}_{\mathrm{new}} = \bar{J}_{\mathrm{fs}} - \Lambda^*(1 - \epsilon)\bar{J}_{\mathrm{old}}$$

Operator Splitting in RT

- solved directly to obtain $\bar{J}_{\rm new}$

$$\bar{J}_{\rm new} = \left[1 - \Lambda^*(1-\epsilon)\right]^{-1} \left(\bar{J}_{\rm fs} - \Lambda^*(1-\epsilon)\bar{J}_{\rm old}\right)$$

Operator Splitting in RT

- mathematically:
- same family as Jacobi or Gauss-Seidel methods
- general form

$$Mx^{k+1} = Nx^k + b$$

for solution of linear system

$$Ax = b$$

with

$$A = M - N$$

Operator Splitting in RT

- operator splitting $\rightarrow$

$$M = 1 - \Lambda^*(1 - \epsilon)$$

and

$$N = (\Lambda - \Lambda^*)(1 - \epsilon)$$

for the system matrix

$$A = 1 - \Lambda(1 - \epsilon)$$

Operator Splitting in RT

- convergence of the iterations $\rightarrow$
- spectral radius $\rho(G) < 1$
- with amplification matrix

$$G = M^{-1}N$$

Operator Splitting in RT

- for this to help
  - eigenvalues of amplification matrix $G \ll 1$
  - works best if $\Lambda^* = \Lambda$ (direct solution, how? see below!)
  - $\rightarrow$ expensive (?)
  - diagonal $\Lambda^* \rightarrow$
    simple but slow convergence
  - band-matrix $\Lambda^* \rightarrow$
    rapid convergence, harder to construct

Operator Splitting in RT

- many possible ways to construct Λ^*
- best: elements of Λ itself
  - no estimates/free parameters
  - 'easy' to compute & use
  - build band-matrix Λ^*

Formal Solution

- pp RTE:

$$\mu \frac{dI}{d\tau_r} = I - S$$

- discretize τ_r (spatial) space
- discretize μ (angle) space
- light path $\rightarrow$ lines of constant μ

Formal Solution

- $\rightarrow$ characteristics of the RTE
- along a characteristic

$$\frac{dI}{d\tau} = I - S$$

- where $I = I(\mu)$ and $\tau = \tau_r/\mu$ are now measured along the characteristic

Formal Solution

- spherical geometry:

$$\mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} = \chi(S - I)$$

- characteristics = light paths
- static medium → straight lines
- parameterized by *impact parameter* p

$$\mu(r) = \pm\sqrt{1 - p^2/r^2}$$

gives μ for each point along the characteristic p

Formal Solution

- $\rightarrow$ along a characteristic μ varies in spherical geometry
- but the RTE along the characteristic looks like

$$\frac{dI}{d\tau} = I - S$$

- where $I = I(\mu(p))$ and τ are now measured along the characteristic

Formal Solution

- $\rightarrow$ spherical and pp cases reduced to FS along the characteristic
- J computed by numerical quadrature
- need to compute I along the characteristics
- basic idea:
  - approximate S along the characteristic by piece-wise linear or parabolic functions
  - analytically solve FS along the characteristic for I

Formal Solution

- this scheme gives

$$
\begin{aligned}
I(\tau_i) &= I(\tau_{i-1})\exp(\tau_{i-1}-\tau_i) + \int_{\tau_{i-1}}^{\tau_i} S(\tau)\exp(\tau-\tau_i)\,d\tau \\
I(\tau_i) &\equiv I_{i-1}\exp(-\Delta\tau_{i-1}) + \Delta I_i
\end{aligned}
$$

- τ_i denotes the optical depth along the ray with $\tau_1 \equiv 0$ and τ_{i-1}
- τ is calculated using piecewise linear interpolation of χ along the ray

$$
\Delta\tau_{i-1} = (\chi_{i-1}+\chi_i)|s_{i-1}-s_i|/2
$$

Formal Solution

- $S(\tau)$ along a characteristic is interpolated by linear or parabolic polynomials so that

$$\Delta I_i = \alpha_i S_{i-1} + \beta_i S_i + \gamma_i S_{i+1}$$

with

$$\begin{aligned}
\alpha_i &= e_{0i} + [e_{2i} - (\Delta\tau_i + 2\Delta\tau_{i-1})e_{1i}]/[\Delta\tau_{i-1}(\Delta\tau_i + \Delta\tau_{i-1})] \\
\beta_i &= [(\Delta\tau_i + \Delta\tau_{i-1})e_{1i} - e_{2i}]/[\Delta\tau_{i-1}\Delta\tau_i] \\
\gamma_i &= [e_{2i} - \Delta\tau_{i-1}e_{1i}]/[\Delta\tau_i(\Delta\tau_i + \Delta\tau_{i-1})]
\end{aligned}$$

for parabolic interpolation and

Formal Solution

$$\begin{aligned}\alpha_i &= e_{0i} - e_{1i}/\Delta\tau_{i-1} \\ \beta_i &= e_{1i}/\Delta\tau_{i-1} \\ \gamma_i &= 0\end{aligned}$$

for linear interpolation.

- auxiliary functions:

$$\begin{aligned}e_{0i} &= 1 - \exp(-\Delta\tau_{i-1}) \\ e_{1i} &= \Delta\tau_{i-1} - e_{0i} \\ e_{2i} &= (\Delta\tau_{i-1})^2 - 2e_{1i}\end{aligned}$$

Formal Solution

- $\Delta\tau_i \equiv \tau_{i+1} - \tau_i$ is the optical depth along the characteristic
- must use linear coefficients at the last integration point along each ray
- some times linear may be better than parabolic (why?)

Formal Solution

- spherical case:
  - tangent rays: FS starts at point 2 with I_1 given as the outer BC and proceeds along the ray
  - FS for 'core-intersecting' ray: split into two parts:
    1. integration from point 1 to point N, where I_1 is given as the outer boundary condition and
    2. integration from point $N+2$ to point $2N$, where I_{N+1} is given as the inner boundary condition.
- pp case: like 'core intersecting' spherical case

Formal Solution

- with this procedure, the I's can be computed for given S
- for all characteristics $\rightarrow$ full RF is known
- could be used for Λ iteration for $\epsilon \approx 1$
- next step: devise procedure to compute Λ^*!
- space-discretized $\rightarrow$
  Λ operator $\rightarrow$ Λ matrix

constructing Λ^*

- basic idea:
  - set $S = (0, \ldots, 1, 0, \ldots)$
  - perform FS
  - $\rightarrow$ delivers one column of Λ matrix
  - repeat for all spatial (radial) points
  - $\rightarrow$ compute Λ matrix
- if done like described $\rightarrow$ very expensive!
- however, it can be done analytically!

constructing Λ^*

- describe spherical case
- pp case $\rightarrow$ just consider 'core intersecting' characteristics
- write

$$\Lambda^* = \Lambda^t + \Lambda^c$$

- Λ^t: contributions from tangential characteristics
- Λ^c: contributions from core intersecting characteristics
- do not construct full Λ matrix (time!)
- construct tri-diagonal Λ^*

constructing Λ^*

- Λ^t construction
- 1 or 2 intersection of tangential characteristics with radial grid points
- label characteristic tangent to shell $i+1$ with the index i
- $\rightarrow$ construction of Λ^t contributions

constructing Λ^*

- (a) $1 < j \leq i$

$$\begin{aligned}
\Lambda^t_{j-1,j} &\mathrel{+}= \sum_i \left(w_{j-1,i} s_{1,i} + w_{k+1,i} s_{6,i} \right) \\
\Lambda^t_{j,j} &\mathrel{+}= \sum_i \left(w_{j,i} s_{2,i} + w_{k,i} s_{5,i} \right) \\
\Lambda^t_{j+1,j} &\mathrel{+}= \sum_i \begin{cases} \left(w_{j+1,i} s_{3,i} + w_{k-1,i} s_{4,i} \right) & \text{for } i \neq j \\ w_{k-1,i} s_{4,i} & \text{for } i = j \end{cases}
\end{aligned}$$

with $k = 2(i-1) - j$ and

constructing Λ^*

$$
\begin{aligned}
s_{1,i} &= \gamma_{j-1,i} \\
s_{2,i} &= s_{1,i} \exp\bigl(-\Delta\tau_{j-1,i}\bigr) + \beta_{j,i} \\
s_{3,i} &= s_{2,i} \exp\bigl(-\Delta\tau_{j,i}\bigr) + \alpha_{j+1,i} \\
s_{4,i} &= s_{3,i} \exp\bigl(-\Delta\tau_{j+1\to k-1,i}\bigr) + \gamma_{k-1,i} \\
s_{5,i} &= s_{4,i} \exp\bigl(-\Delta\tau_{k-1,i}\bigr) + \beta_{k,i} \\
s_{6,i} &= s_{5,i} \exp\bigl(-\Delta\tau_{k,i}\bigr) + \alpha_{k+1,i}
\end{aligned}
$$

constructing Λ^*

- (b) $j = 1$

$$\Lambda^t_{1,1} \mathrel{+}= \sum_i \left(w_{1,i} s_{2,i} + w_{k,i} s_{5,i} \right)$$

$$\Lambda^t_{2,1} \mathrel{+}= \sum_i \begin{cases} \left(w_{2,i} s_{3,i} + w_{k-1,i} s_{4,i} \right) & \text{for } i \neq 1, \\ w_{k-1,i} s_{4,i} & \text{for } i = 1, \end{cases}$$

with $k = 2i + 1$ and

constructing Λ^*

$$
\begin{aligned}
s_{2,i} &= \beta_{1,i} \\
s_{3,i} &= s_{2,i} \exp(-\Delta\tau_{1,i}) + \alpha_{2,i} \\
s_{4,i} &= s_{3,i} \exp(-\Delta\tau_{2\rightarrow k-1,i}) + \gamma_{k-1,i} \\
s_{5,i} &= s_{4,i} \exp(-\Delta\tau_{k-1,i}) + \beta_{k,i} \\
s_{6,i} &= s_{5,i} \exp(-\Delta\tau_{k,i}) + \alpha_{k+1,i}
\end{aligned}
$$

constructing Λ^*

- (c) $j = i + 1$ (point of tangency)

$$\begin{aligned} \Lambda^t_{i,i+1} &\mathrel{+}= \sum_i \left(w_{i,i} s_{1,i} + w_{i+2,i} s_{3,i} \right) \\ \Lambda^t_{i+1,i+1} &\mathrel{+}= \sum_i w_{i,i} s_{2,i} \end{aligned}$$

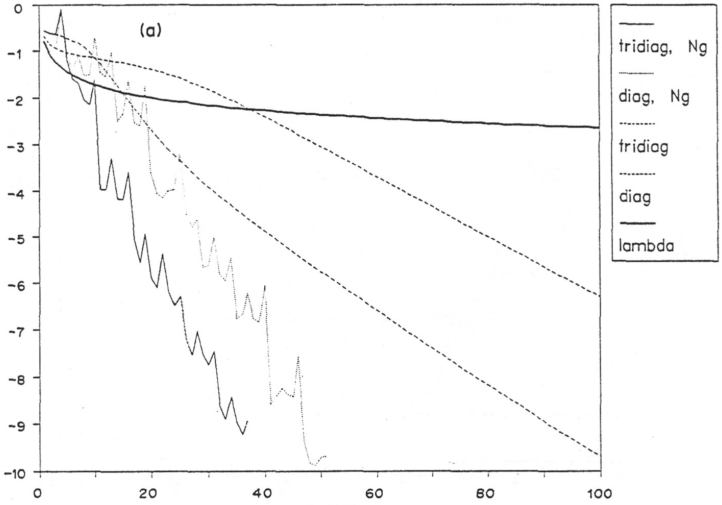
constructing Λ^*

$$
\begin{aligned}
s_{1,i} &= \gamma_{i,i} \\
s_{2,i} &= s_{1,i} \exp(-\Delta\tau_{i,i}) + \beta_{i+1,i} \\
s_{3,i} &= s_{2,i} \exp(-\Delta\tau_{i+1,i}) + \alpha_{i+1,i}
\end{aligned}
$$

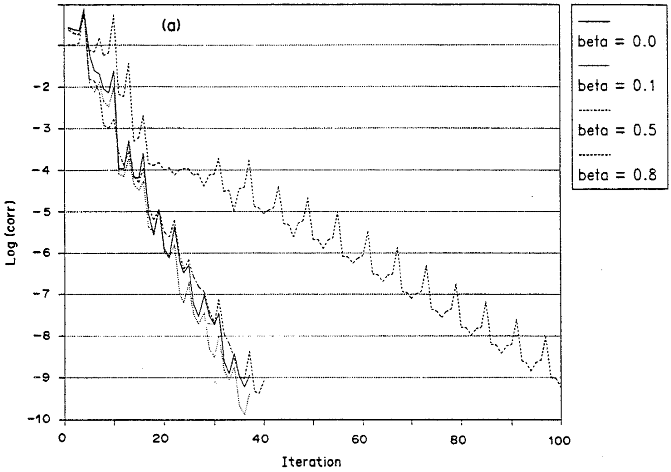
constructing Λ^*

- $\Delta\tau_{j+1\to k-1,i}$: optical depth along the characteristic
- $w_{j,i}$: μ quadrature weights
- can be extended to full Λ matrix!
- core intersecting characteristics: simplified version of above
- this method can be used also in moving media
- can be extended to line transfer (later)

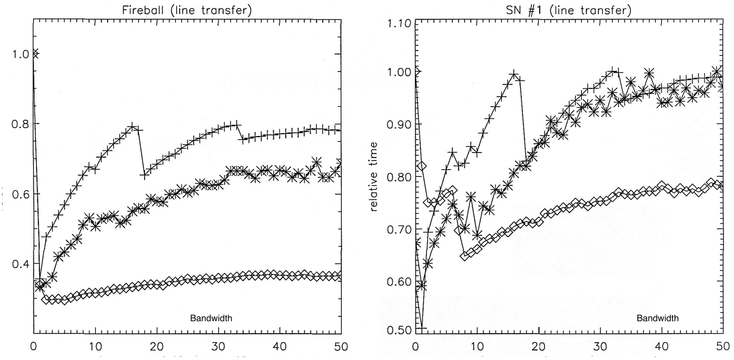
Convergence: static



Convergence: expanding

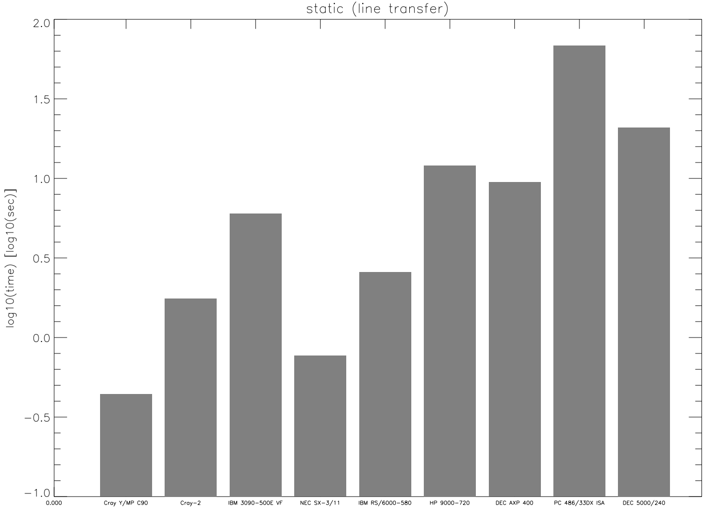


relative Performance



- SN: $v_{\max} = 0.13c$, extension 100
- Fireball: $v_{\max} = 0.9c$, extension 10^6
- $+$: Cray Y/MP C90, $*$: Cray-2, other: IBM 3090-500E VF

relative Performance



- historical ... ca. 1993