Stellar/Planetary Atmospheres

Part 03: grey atmosphere

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14. März 2018

Topics

- The grey atmosphere
  - Milne's equation
  - opacity means
  - approximate solution

grey atmosphere

- grey approximation
  - assume wavelength independent extinction coefficient

$$\chi_\nu = \chi = \text{const. in } \nu$$

- not really realistic but
  1. non-grey problem can be reduced to gray case with opacity averages
  2. useful baseline for analyses of non-grey problems

grey atmosphere

- consider grey atmosphere in RE!
- pp RTE

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

- reduces in the grey approximation to

$$\mu \frac{dI}{d\tau} = I - S$$

grey atmosphere

- where

$$
\begin{aligned}
I &= \int_0^\infty I_\nu \, d\nu \\
\tau &= \int_0^\infty \tau_\nu \, d\nu
\end{aligned}
$$

and so on

grey atmosphere

- the condition of RE is

$$\int_0^\infty \chi_\nu S_\nu \, d\nu = \int_0^\infty \chi_\nu J_\nu \, d\nu$$

- this implies in the grey atmosphere

$$S = J$$

- $\rightarrow$ the grey RTE in RE is

$$\mu \frac{dI}{d\tau} = I - J$$

grey atmosphere

- $\rightarrow$ formal solution

$$J(\tau) = \Lambda_\tau[S(t)] = \frac{1}{2}\int_0^\infty E_1(|t-\tau|)J(t)\,dt$$

- *Milne's Equation*
- solution will automatically satisfy the grey RTE and the RE condition

grey atmosphere

- if we introduce LTE
$$S_\nu = B_\nu$$
- $\rightarrow$ from the RE condition
$$J(\tau) = S(\tau) = B(T(\tau)) = \frac{\sigma}{\pi} T^4(\tau)$$
- $\rightarrow$ solution of grey RTE associates a *temperature structure* with the RE condition!

grey atmosphere

- integrating the RTE over $\mu \rightarrow$

$$\frac{dH}{d\tau} = J - S = J - J = 0$$

- first moment $(1/2 \int \mu f(\mu)\, d\mu)$ of the RTE:

$$2\frac{d}{d\tau}\int \mu^2 I\, d\mu = \int \mu I\, d\mu - \int \mu S\, d\mu = 2H - S\int \mu\, d\mu = 2H$$

- so that

$$\frac{dK}{d\tau} = H = \mathrm{const.}$$

grey atmosphere

- with the exact integral

$$K(\tau) = H\tau + \mathrm{const.} = \frac{1}{4}F\tau + c$$

grey atmosphere

- A relation between J and K follows from the Eddington relation
$$J \to 3K \; (\tau \to \infty)$$
- very good at large τ!
- with
$$K(\tau) \to \frac{1}{4} F\tau \; (\tau \to \infty)$$
we have
$$J(\tau) \to \frac{3}{4} F\tau \; (\tau \to \infty)$$
- at large τ, $J(\tau)$ will be linear in τ.

grey atmosphere

- at small τ: $J(\tau)$ will deviate from this!
- $\rightarrow$ explicitly split out linear part:

$$J(\tau) = \frac{3}{4} \left(\tau + q(\tau)\right) F$$

where $q(\tau)$ is the *Hopf-function*

grey atmosphere

- $q(\tau)$ at large τ

$$0 = \lim_{\tau\to\infty}\left(\frac{1}{3}J(\tau) - K(\tau)\right) = \frac{1}{4}F\lim_{\tau\to\infty}(\tau + q(\tau) - \tau - c)$$

- therefore

$$q(\infty) = c$$

- so that

$$K(\tau) = \frac{1}{4}(\tau + q(\infty))F$$

grey atmosphere

- the solution of the grey problem is now reduced to finding $q(\tau)$
- insert

$$J(\tau) = \frac{3}{4}(\tau + q(\tau))F$$

into the formal solution

$$J(\tau) = \frac{1}{2}\int_0^\infty E_1(|t-\tau|)J(t)\,dt$$

grey atmosphere

- this gives

$$\tau + q(\tau) = \frac{1}{2}\int_0^\infty E_1(|t-\tau|)(\tau + q(\tau))\,dt$$

- and for $T(\tau)$ we have

$$T^4(\tau) = \frac{3}{4}T_{\mathrm{eff}}^4(\tau + q(\tau))$$

Mean opacities

- before going through the trouble to find exact and approximate solutions, let's look into connections between grey and non-grey cases
- it is possible to reduce a non-grey to a grey problem
- compare moment equations

$$\mu \frac{dI_\nu}{dz} = \chi_\nu (S_\nu - I_\nu)$$
$$\mu \frac{dI}{dz} = \chi (S - I)$$

Mean opacities

$$
\begin{aligned}
\frac{dH_\nu}{dz} &= \chi_\nu (S_\nu - J_\nu) \\
\frac{dH}{dz} &= \chi (S - J)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dK_\nu}{dz} &= -\chi_\nu H_\nu \\
\frac{dK}{dz} &= -\chi H
\end{aligned}
$$

flux weighted mean

- try to find mean opacity $\bar{\chi}$ so that

$$\frac{dK_\nu}{dz} = -\chi_\nu H_\nu$$

assumes the form

$$\frac{dK}{dz} = -\bar{\chi} H$$

when integrated over ν

- in that case,

$$K(\bar{\tau}) = H\bar{\tau} + c$$

will be exact also in the non-grey case!

flux weighted mean

- integrate
$$\frac{dK_\nu}{dz} = -\chi_\nu H_\nu$$
over $\nu \rightarrow$
$$-\int_0^\infty \frac{dK_\nu}{dz}\, d\nu = -\frac{dK}{dz} = \int_0^\infty \chi_\nu H_\nu\, d\nu = \bar{\chi}_F H$$

flux weighted mean

- which leads to the definition of the *flux weighted mean opacity*

$$\bar{\chi}_F = \frac{\int_0^\infty \chi_\nu H_\nu \, d\nu}{H}$$

- and

$$\frac{dK}{dz} = -\bar{\chi}_F H$$

flux weighted mean

- problems:
  1. we need to know H_ν to compute $\bar{\chi}_F$
  2. the other mono-chromatic eqs. do *not* transfer into their grey counterparts
- but: it recovers the correct value of the radiation pressure $P_{\rm rad} = (4\pi/c)K$

flux weighted mean

- therefore it also recovers the correct radiation force

$$\frac{dP_{\rm rad}}{dz} = \frac{1}{\bar{\chi}_F}\frac{dP_{\rm rad}}{d\bar{\tau}} = \frac{4\pi}{c\chi_F}\int_0^\infty \chi_\nu H_\nu \, d\nu = \frac{4\pi}{c}H = \frac{\sigma}{c}T_{\rm eff}^4$$

- this gives a simple expression for $dP_{\rm rad}/dz$ if $\bar{\chi}_F$ is known.

Rosseland mean

- construct average so that correct value of frequency integrated flux is recovered

$$H = \int_0^\infty H_\nu \, d\nu = -\int_0^\infty \frac{1}{\chi_\nu} \frac{dK_\nu}{dz} \, d\nu \equiv -\frac{1}{\bar{\chi}} \frac{dK}{dz}$$

- therefore

$$\frac{1}{\bar{\chi}} = \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{dK_\nu}{dz} \, d\nu}{\int_0^\infty \frac{dK_\nu}{dz} \, d\nu}$$

Rosseland mean

- at large τ we must have

$$K_\nu \rightarrow \frac{1}{3} J_\nu$$

and

$$J_\nu \rightarrow B_\nu$$

so that we define

$$\frac{1}{\bar{\chi}_R} = \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{dB_\nu}{dT} \, d\nu}{\int_0^\infty \frac{dB_\nu}{dT} \, d\nu}$$

- *Rosseland mean opacity*

Rosseland mean

- these assumptions are the same as made earlier for the diffusion approximation!
- $\rightarrow$
$$H_\nu = -\frac{1}{3}\frac{1}{\chi_\nu}\frac{dB}{dT}\frac{dT}{dz}$$
is transformed into
$$H = -\frac{1}{3}\frac{1}{\bar{\chi}_R}\frac{dB}{dT}\frac{dT}{dz}$$

Rosseland mean

- at large τ we have

$$T^4(\bar{\tau}_R) = \frac{3}{4} T_{\mathrm{eff}}^4(\bar{\tau}_R + q(\bar{\tau}_R))$$

  is a good approximation even in the non-grey case!
- diffusion approximation breaks down closer to the surface!
- $\rightarrow$ flux conservation not guaranteed if Rosseland mean is used close to surface!

Planck & absorption means

- defined to yield correct value of the thermal emission:

$$\int \kappa_\nu B_\nu \, d\nu = \bar{\kappa}_P \int B_\nu \, d\nu = \frac{\sigma}{\pi} T^4 \bar{\kappa}_P$$

- *Planck absorption mean*

Planck & absorption means

- analogous: define *absorption mean*

$$\bar{\kappa}_J J = \int \kappa_\nu J_\nu \, d\nu$$

  which gives correct total amount of energy absorbed in the medium

- both means do not lead to further simplifications in the RTE but are useful in rad-hydro calculations

approximate grey solution

- for $\tau \gg 1$ we have

$$J(\tau) = 3K(\tau)$$

- this relation is valid for some other special cases:
- look at $\tau = 0$
- $I(\mu \leq 0) = 0$
- $I(\mu \geq 0) = I_0 = \mathrm{const.}$

approximate grey solution

- with this

$$\begin{aligned} J &= \frac{1}{2}\int I\,d\mu = \frac{1}{2}I_0\int_0^1 d\mu = \frac{1}{2}I_0 \\ K &= \frac{1}{2}\int \mu^2 I\,d\mu = \frac{1}{2}I_0\int_0^1 \mu^2\,d\mu = \frac{1}{6}I_0 \end{aligned}$$

- $\rightarrow J = 3K$

approximate grey solution

- also *two-stream approximation*:
- $I(\mu \leq 0) = I_-$
- $I(\mu \geq 0) = I_+ = \mathrm{const.}$
- with this

$$
\begin{aligned}
J &= \frac{1}{2} I_+ \int_0^1 d\mu + \frac{1}{2} I_- \int_{-1}^0 d\mu = \frac{1}{2}(I_+ + I_-) \\
K &= \frac{1}{2} I_+ \int_0^1 \mu^2 \, d\mu + \frac{1}{2} I_- \int_{-1}^0 \mu^2 \, d\mu = \frac{1}{6}(I_+ + I_-)
\end{aligned}
$$

- $\rightarrow J = 3K$

approximate grey solution

- Eddington $\rightarrow$ assume that $J = 3K$ everywhere!
- $\rightarrow$
$$J(\tau) = \frac{3}{4} F\tau + c'$$
- to compute c', insert this into the FS for $F(0)$:

$$\begin{aligned} F(0) &= 2\int_0^\infty E_2(t)\left\{\frac{3}{4}F\tau + c'\right\} dt \\ &= 2c' E_3(0) + \frac{3}{4}F\left\{\frac{4}{3} - 2E_4(0)\right\} \end{aligned}$$

approximate grey solution

- set $F(0) = F_0$ to the target flux
- for the exponential integrals we have the relation

$$E_n(0) = \frac{1}{n-1}$$

so that $c' = F_0/2$

- therefore

$$\begin{aligned} J(\tau) &= \frac{3}{4} F_0 \left(\tau + \frac{2}{3}\right) \\ T^4(\tau) &= \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3}\right) \end{aligned}$$

approximate grey solution

- boundary temperature

$$T(0) = \left(\frac{1}{2}\right)^{1/4} T_{\mathrm{eff}} \approx 0.841\, T_{\mathrm{eff}}$$

agrees well with exact value

$$T(0) = \left(\frac{\sqrt{3}}{4}\right)^{1/4} T_{\mathrm{eff}} \approx 0.84114\, T_{\mathrm{eff}}$$

limb darkening

- insert $J(\tau)$ from the Eddington approx into the FS to compute angular dependence of I at $\tau = 0$

$$
\begin{aligned}
I(0,\mu) &= \frac{3}{4}F_0 \int_0^{\infty} \left(\tau + \frac{2}{3}\right) \frac{\exp(-\tau/\mu)}{\mu}\, dt \\
&= \frac{3}{4}F_0 \left(\mu + \frac{2}{3}\right)
\end{aligned}
$$

- specific form of Eddington-Barbier relation!
- *limb darkening law*: ratio

$$
\frac{I(0,\mu)}{I(0,\mu = 1)}
$$

limb darkening

- in the Eddington approx:

$$\frac{I(0,\mu)}{I(0,\mu=1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right)$$

- so that

$$\frac{I(0,0)}{I(0,1)} = \frac{2}{5} = 0.4$$

- $\rightarrow$ good agreement with measured value for the Sun!

exact solution

- basic idea:
  - write RTE in the form

$$\mu \frac{dI}{d\tau} = I - \frac{1}{2} \int I \, d\mu$$

  - approximate $\int d\mu$ by quadrature sum

$$\frac{1}{2} \int I \, d\mu = \sum_{-n}^{n} a_j I(\mu_j)$$

exact solution

- insert this into the RTE:

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{1}{2} \sum_{-n}^{n} a_j I(\mu_j)$$

this is a linear, first order ODE, solutions of the form

$$I(\mu_i) = g_i \exp(-k_i \tau)$$

are good guesses for test solutions

exact solution

- inserting the test functions into the RTE:

$$g_i(1 + k_i \mu_i) = \frac{1}{2} \sum_{-n}^{n} a_j g_j = \mathrm{const} = c$$

- so that

$$g_i = \frac{c}{1 + k_i \mu_i}$$

- inserting that back into the RTE
  $\rightarrow$ characteristic equation

$$\frac{1}{2} \sum_{-n}^{n} \frac{a_j}{1 + k_j \mu_j} = 1$$

exact solution

- solving this and computing its roots gives
  → *general solution of the RTE*

$$
\begin{aligned}
I_i(\tau) &= b\left\{ \sum_{\alpha=1}^{n} L_\alpha \left(1 + k_\alpha \mu_i\right)^{-1} \exp(-k_\alpha \tau) \right. \\
&\qquad \left. + \sum_{\alpha=1}^{n-1} L_{-\alpha} \left(1 - k_\alpha \mu_i\right)^{-1} \exp(+k_\alpha \tau) \right\}
\end{aligned}
$$

exact solution

- particular solution:

$$I_i(\tau) = b(\tau + Q + \mu_i)$$

- $\rightarrow$ complete solution

$$\begin{aligned} I_i(\tau) &= b\left\{\tau + Q + \mu_i + \sum_{\alpha=1}^{n} L_\alpha \left(1 + k_\alpha \mu_i\right)^{-1} \exp(-k_\alpha \tau) \right. \\ &\quad \left. + \sum_{\alpha=1}^{n-1} L_{-\alpha} \left(1 - k_\alpha \mu_i\right)^{-1} \exp(+k_\alpha \tau) \right\} \end{aligned}$$

- with $2n$ constants b, Q, L_α and $L_{-\alpha}$

exact solution

- constants can be computed by applying the boundary conditions
- $\rightarrow L_{-\alpha} = 0$ (inner BC)
- remaining constants $\rightarrow$ linear system

$$Q - \mu_i + \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - k_\alpha \mu_i} = 0$$

- discrete ordinate representation of the Hopf function:

$$q(\tau) = Q + \sum_{\alpha=1}^{n-1} L_\alpha \exp(-k_\alpha \tau)$$

exact solution

- For $n = 1$ and 2 we get for q:
- $n = 1$

$$q(\tau) = \frac{1}{\sqrt{3}}$$

- $n = 2$

$$q(\tau) = 0.694025 - 0.116675 \exp(-1.97203\tau)$$

- exact solution can be obtained by $n \to \infty$