Stellar/Planetary Atmospheres

Part 02: equation of transfer

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Topics

- The equation of radiative transfer
  - kinetic form
  - source function
  - with scattering
  - RTE in different geometries
  - RTE in moving atmospheres
  - optical depth in depth
  - formal solution of the RTE
  - Eddington-Barbier relation
  - Schwarzschild-Milne equations
  - radiative equilibrium
  - diffusion approximation

RTE

- Outline
  1. introduce 'kinetic equation' for particles
  2. look at the corresponding equation of photons
  3. rewrite it into the conventional form

kinetic equation for particles

$$\frac{\partial \Phi}{\partial t} = -\nabla^{\mu} \cdot (\vec{x}\Phi) + \left(\frac{\partial \Phi}{\partial t}\right)_{\mathrm{coll}}$$

- Φ: particle distribution function
- $\nabla^{\mu} \cdot (\vec{x}\Phi)$: divergence of the phase space particle flow
- Gauss' theorem $\rightarrow=$ net flow of particles through a phase space volume element

kinetic equation for particles

$$
\begin{aligned}
\nabla^{\mu} \cdot (\vec{x}\Phi) &= \frac{\partial(\dot{\vec{r}}\Phi)}{\partial \vec{r}} + \frac{\partial(\dot{\vec{v}}\Phi)}{\partial \vec{v}} \\
&= \frac{\partial(\vec{v}\Phi)}{\partial \vec{r}} + \frac{\partial}{\partial \vec{v}}\left(\frac{\vec{K}}{m}\Phi\right) \\
&= \vec{v} \cdot \frac{\partial \Phi}{\partial \vec{r}} + \frac{\vec{K}}{m} \cdot \frac{\partial \Phi}{\partial \vec{v}}
\end{aligned}
$$

- $\vec{K}$: force
- assumes that $\vec{K}$ is of the form

$$
\vec{K} = m\vec{g} + q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}
$$

kinetic equation for particles

- with this we get

$$\frac{\partial \Phi}{\partial t} + \vec{v} \cdot \frac{\partial \Phi}{\partial \vec{r}} + \frac{\vec{K}}{m} \cdot \frac{\partial \Phi}{\partial \vec{v}} = \left(\frac{\partial \Phi}{\partial t} \right)_{\mathrm{coll}}$$

- $\left(\frac{\partial \Phi}{\partial t}\right)_{\mathrm{coll}}$: collisional term
- strictly speaking: defined by the above equation

kinetic equation for particles

- Properties of $\left(\frac{\partial \Phi}{\partial t}\right)_{\rm coll}$:
  1. depends only on one-particle distribution function, no correlation effects
  2. depends only on instantaneous values of Φ and $\partial \Phi/\partial t$, not on the history of the system (Markov-process)
  3. leads irreversibly to TE
- note: particles interact also through the force term!

kinetic equation for photons

$$\frac{\partial \Phi}{\partial t} + c\vec{n} \cdot \frac{\partial \Phi}{\partial \vec{r}} = \left(\frac{\partial \Phi}{\partial t}\right)_+ - \left(\frac{\partial \Phi}{\partial t}\right)_-$$

- $\left(\frac{\partial \Phi}{\partial t}\right)_+$: creation of photons
- $\left(\frac{\partial \Phi}{\partial t}\right)_-$: destruction of photons
- main difference to particles: no force term

kinetic equation for photons

- with $\Phi = c^2/(h^4\nu^3) I_\nu \rightarrow$

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{n}\cdot\nabla I_\nu = \frac{1}{c}\left(\frac{\partial I_\nu}{\partial t}\right)_+ - \frac{1}{c}\left(\frac{\partial I_\nu}{\partial t}\right)_-$$

- photon creation usually written as

$$e_\nu = \frac{1}{c}\left(\frac{\partial I_\nu}{\partial t}\right)_+$$

- destruction is (here) always $\propto I_\nu \rightarrow$

$$a_\nu I_\nu = \frac{1}{c}\left(\frac{\partial I_\nu}{\partial t}\right)_-$$

kinetic equation for photons

- with
$$\frac{1}{c}\frac{\partial I_\nu}{\partial t}\, d\nu d\Omega = h\nu \frac{\partial \Phi}{\partial t} d^3p$$
we have:
  - $e_\nu\, d\nu d\Omega$: $h\nu$ times the creation rate of photons in $(\nu, d\nu, \vec{n}, d\Omega)$
  - $a_\nu I_\nu\, d\nu d\Omega$: $h\nu$ times the destruction rate of photons in $(\nu, d\nu, \vec{n}, d\Omega)$

absorption/emission of photons

- photon creation is always composed of spontaneous (ϵ_ν) and stimulated parts:

$$e_\nu = \left(1 + \frac{c^2}{2h\nu^3} I_\nu\right) \epsilon_\nu$$

absorption/emission of photons

- thus define

$$
\begin{aligned}
e_\nu - a_\nu I_\nu &= \epsilon_\nu + \epsilon_\nu \frac{c^2}{2h\nu^3} I_\nu - a_\nu I_\nu \\
&= \epsilon_\nu - \left(a_\nu - \epsilon_\nu \frac{c^2}{2h\nu^3} \right) I_\nu \\
&\equiv \epsilon_\nu - \chi_\nu I_\nu
\end{aligned}
$$

- ϵ_ν: emission coefficient
- χ_ν: extinction coefficient
- e_ν & a_ν describe the true physical creation and destruction of photons (QED)
- ϵ_ν & χ_ν account for the net rate of change of photons

The source function

- the ratio ϵ_ν/χ_ν is called the *source function*

$$S_\nu = \frac{\epsilon_\nu}{\chi_\nu}$$

- translated from 'Ergiebigkeit'
- re-translated as 'Quellfunktion'

The RTE

- with this we can write the radiative transfer equation (RTE) in its usual form:

$$\begin{aligned}\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{n}\cdot\nabla I_\nu &= \epsilon_\nu - \chi_\nu I_\nu \\ &= \chi_\nu(S_\nu - I_\nu)\end{aligned}$$

S_ν in TE

- In TE we have
  - $I_\nu = B_\nu$
  - $\partial B_\nu / \partial t = 0$
  - $\nabla I_\nu = \nabla B_\nu = 0$
- therefore $\epsilon_\nu = \chi_\nu I_\nu = \chi_\nu B_\nu$
- so that in TE

$$S_\nu = B_\nu$$

- frequently used in LTE, too!

S_ν & scattering

- stellar/planetary atmosphere gas will absorb and scatter photons
- for simplicity: isotropic, coherent scattering
- extinction given by

$$\chi_\nu = \kappa_\nu + \sigma_\nu$$

- κ_ν: true absorption (destruction)
- σ_ν: scattering
- absorptive processes lead in TE to equivalent emission $\rightarrow$

$$\epsilon_\nu(\mathrm{thermal}) = \kappa_\nu B_\nu$$

S_ν & scattering

- for scattering processes we have

$$\begin{aligned}\epsilon_\nu(\text{scattering}) &= \sigma_\nu \int p(\vec{n}, \vec{n'}) I_\nu \, d\Omega \\ &= \frac{\sigma_\nu}{4\pi} \int I_\nu \, d\Omega \\ &= \sigma_\nu \frac{1}{2} \int I_\nu \, d\mu \quad \text{(1D)} \\ &= \sigma_\nu J_\nu\end{aligned}$$

with

  - p: phase function (isotropic by previous assumption)

S_ν & scattering

- so that we have

$$S_\nu = (1-\epsilon)J_\nu + \epsilon B_\nu$$

with the definition of the *photon destruction probability*

$$\epsilon = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}$$

RTE in different geometries

- Cartesian coordinates (ν index omitted)

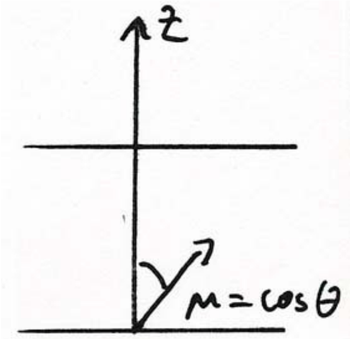
$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{n} \cdot \nabla I = \chi(S - I)$$

- translate $\vec{n} \cdot \nabla$ to obtain RTE in different geometries
- simple cases: time independent

RTE in different geometries

- plane parallel (slab) geometry:

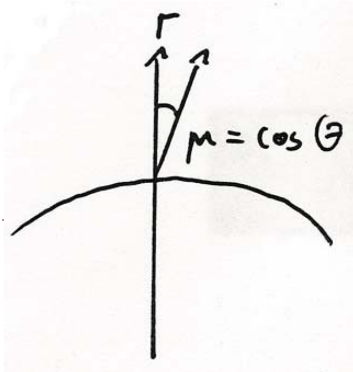
$$\mu \frac{dI}{dz} = \chi(S - I)$$



RTE in different geometries

- spherical geometry:

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I}{\partial \mu} = \chi(S$$



RTE in different geometries

- 1D, spherical, relativistic (Lagrange frame):

$$e\frac{\partial I}{\partial r} + \frac{\partial}{\partial \mu}(fI) + g\frac{\partial}{\partial \lambda}(\lambda I) + hI = \eta - \chi I$$

with

$$\begin{aligned}
e(r,\mu) &= \gamma(\mu+\beta) \\
f(r,\mu) &= \gamma(1-\mu^2)\left[\frac{1+\beta\mu}{r} - \gamma^2(\mu+\beta)\frac{\partial\beta}{\partial r}\right] \\
g(r,\mu) &= \gamma\left[\frac{\beta(1-\mu^2)}{r} + \gamma^2\mu(\mu+\beta)\frac{\partial\beta}{\partial r}\right] \\
h(r,\mu) &= \gamma\left[\frac{\beta(1-\mu^2)}{r} + \gamma^2(1+\mu^2+2\beta\mu)\frac{\partial\beta}{\partial r}\right]
\end{aligned}$$

RTE in different geometries

- $I(r, \mu, \lambda)$: specific intensity scaled by r^2,
- r: radial coordinate,
- μ: cosine of the direction angle, $\mu = \cos\phi$
- v: velocity, $\beta = v/c$, $\gamma^2 = 1/(1-\beta^2)$,
- $\chi(r, \lambda)$: extinction coefficient, $\chi = \kappa + \sigma_e + \kappa_l \varphi(\lambda)$
- $\eta(r, \lambda)$: emissivity.

RTE in different geometries

Example for $\eta(r, \lambda)$

$$\eta = \kappa B_\lambda(T) + \sigma_e J(\lambda) + \kappa_l \varphi(\lambda) \int_0^\infty \varphi(\lambda) J(\lambda) \, d\lambda$$

with

$$J(\lambda) = \int_{-1}^{1} I(\lambda) \, d\mu$$

- $\kappa B_\lambda(T)$: thermal emission
- $\sigma_e J(\lambda)$: electron scattering
- $\frac{\sigma}{2} \int_0^\infty \int_{-1}^{1} \varphi(\lambda) I \, d\mu \, d\lambda$: spectral line emissivity

The optical depth

- pp RTE:

$$\mu \frac{dI}{dz} = \chi(S - I)$$

- we define the optical depth τ by

$$d\tau = -\chi \, dz$$

- with this the RTE becomes

$$\mu \frac{dI}{d\tau} = I - S$$

- total optical depth between 2 points:

$$\Delta\tau_{12} = -\int_{z1}^{z2} \chi(z') \, dz'$$

- $\Delta\tau \ll 1$: optically thin
- $\Delta\tau \gg 1$: optically thick

The optical depth

- along a path s the intensity varies according to

$$dI = -\chi I \, ds + \eta \, ds$$

- if $\eta = 0$, $\chi = \mathrm{const.}$ we have with $d\tau = \chi \, ds$

$$dI = -I \, d\tau$$

so that

$$I(s) = I(s_0) \exp(-\tau)$$

- $\Delta\tau = 1$ is the e-folding optical depth of the intensity in pure extinction

Formal solution of the pp RTE

- PP RTE is a linear first order ODE with constant coefficients
$$\mu \frac{dI}{d\tau} = I - S$$
- integrating factor is $\exp(-\tau/\mu)$:
$$\frac{d(I \exp(-\tau/\mu))}{d\tau} = -\frac{S \exp(-\tau/\mu)}{\mu}$$
- with this$\rightarrow$
$$I \exp\left(-\frac{\tau}{\mu}\right)\Big|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} S(t) \exp\left(-\frac{t}{\mu}\right) \frac{dt}{\mu}$$

Formal solution of the pp RTE

- $\rightarrow$ formal solution

$$
\begin{aligned}
I(\tau_1, \mu) &= I(\tau_2, \mu) \exp\left(-\frac{\tau_2 - \tau_1}{\mu}\right) \\
&\quad + \frac{1}{\mu} \int_{\tau_1}^{\tau_2} S(t) \exp\left(-\frac{t - \tau_1}{\mu}\right) \, dt
\end{aligned}
$$

- formal because it is implicit in S:

$$
S = (1 - \epsilon) J + \epsilon B
$$

Example: semi-infinite slab

- boundary conditions:
  1. outer: $I(\tau = 0, \mu \leq 0) = 0$ no incident radiation
  2. inner:
     - finite $\tau_{\max}$: $I(\tau = \tau_{\max}, \mu \geq 0) = I_{\rm bc}^{+}$
     - $\tau_{\max} \to \infty$:

$$\lim_{\tau \to \infty} I(\tau, \mu \geq 0) \exp\left(-\frac{\tau}{\mu}\right) = 0$$

Example: semi-infinite slab

- for $\mu \geq 0$ we set $\tau_1 = \tau,\ \tau_2 = \infty \rightarrow$

$$I(\tau, \mu \geq 0) = \int_{\tau}^{\infty} S(t) \exp\left(-\frac{t-\tau}{\mu}\right) \frac{dt}{\mu}$$

- for $\mu \leq 0$ we set $\tau_2 = 0,\ I_{\rm bc}^{-} = 0 \rightarrow$

$$I(\tau, \mu \leq 0) = \int_{0}^{\tau} S(t) \exp\left(-\frac{t-\tau}{(-|\mu|)}\right) \frac{dt}{(-|\mu|)}$$

- and for $\tau = 0,\ \mu \geq 0$ (emerging radiation):

$$I(0, \mu \geq 0) = \int_{0}^{\infty} S(t) \exp\left(-\frac{t}{\mu}\right) \frac{dt}{\mu}$$

Eddington-Barbier relation

- linear $S(\tau) = S_0 + S_1\tau \rightarrow$

$$
\begin{aligned}
I(0, \mu \geq 0) &= \int_0^\infty S(t) \exp\left(-\frac{t}{\mu}\right) \frac{dt}{\mu} \\
&= \frac{1}{\mu}\left[S_0 \int_0^\infty \exp\left(-\frac{t}{\mu}\right) dt \right. \\
&\quad \left. + S_1 \int_0^\infty t \exp\left(-\frac{t}{\mu}\right) dt\right] \\
&= S_0 + S_1\mu
\end{aligned}
$$

- $\rightarrow$ Eddington-Barbier relation

$$
I(0, \mu \geq 0) = S_0 + S_1\mu = S(\tau = \mu)
$$

Eddington-Barbier relation

- emergent intensity is characteristic of the value of S at $\Delta\tau = 1$ along the path of the light ray!
- widely used in early analyses of stellar spectra
- not valid in reality
  1. $S(\tau)$ is not linear
  2. scattering $\rightarrow$ I contains contributions from larger (optical) depth range

Schwarzschild-Milne equations

- integrate FS over μ:

$$
\begin{aligned}
J(\tau) &= \frac{1}{2}\int_{-1}^{+1} I \, d\mu \\
&= \frac{1}{2}\left[\int_0^1 d\mu \int_\tau^\infty S(t) \exp\left(-\frac{t-\tau}{\mu}\right) \frac{dt}{\mu}\right. \\
&\quad \left. + \int_{-1}^0 d\mu \int_0^\tau S(t) \exp\left(-\frac{\tau - t}{\mu}\right) \frac{dt}{(-|\mu|)}\right] \\
&= \frac{1}{2}\left\{\int_\tau^\infty S(t) \left[\int_1^\infty \frac{1}{w} \exp\left(-w(t-\tau)\right) \, dw\right] dt \right. \\
&\quad \left. + \int_0^\tau S(t) \left[\int_1^\infty \frac{1}{w} \exp\left(-w(\tau - t)\right) \, dw\right] dt \right\}
\end{aligned}
$$

- with $w = \pm 1/\mu$

Schwarzschild-Milne equations

- [...] is a standard form form of the 1^{st} exponential integral:

$$E_n(x) = \int_1^\infty t^{-n} \exp(-xt)\, dt$$

- using E_1 we have

$$\begin{aligned} J(\tau) &= \frac{1}{2}\left\{ \int_\tau^\infty E_1(t-\tau) S(t)\, dt + \int_0^\tau E_1(\tau - t) S(t)\, dt \right\} \\ &= \frac{1}{2} \int_0^\infty E_1(|t-\tau|) S(t)\, dt \end{aligned}$$

Schwarzschild-Milne equations

- fundamental importance for RT theory $\rightarrow$
- introduce operator notation

$$
\begin{aligned}
\Lambda_\tau[f(t)] &= \frac{1}{2}\int_0^\infty E_1(|t-\tau|)f(t)\,dt \\
J(\tau) &= \Lambda_\tau[S]
\end{aligned}
$$

Schwarzschild-Milne equations

- similar analysis for $F(\tau)$ and $K(\tau) \rightarrow$

$$
\begin{aligned}
F(\tau) &= 2\left\{\int_{\tau}^{\infty} E_2(t-\tau)S(t)\,dt - \int_0^{\tau} E_2(\tau-t)S(t)\,dt\right\} \\
K(\tau) &= \frac{1}{2}\int_0^{\infty} E_3(|t-\tau|)S(t)\,dt
\end{aligned}
$$

- and define the operators

$$
\begin{aligned}
\Phi_\tau[f(t)] &= 2\left\{\int_{\tau}^{\infty} E_2(t-\tau)f(t)\,dt - \int_0^{\tau} E_2(\tau-t)f(t)\,dt\right\} \\
X_\tau[f(t)] &= \frac{1}{2}\int_0^{\infty} E_3(|t-\tau|)f(t)\,dt
\end{aligned}
$$

radiative equilibrium

- assume all energy transported by radiation $\rightarrow$

$$\frac{dF}{dz} = 0$$

- so that

$$F = F_0 = \sigma T_{\mathrm{eff}}^4 = \mathrm{const.}$$

- with

$$F = \int_0^\infty F_\nu \, d\nu$$

radiative equilibrium

- integrate pp RTE

$$\mu \frac{dI_\nu}{dz} = \chi_\nu (S_\nu - I_\nu)$$

over $d\Omega d\nu \rightarrow$

$$\frac{d}{dz} \int_{4\pi} \mu I \, d\Omega = \int_0^\infty \chi_\nu \left\{ \int_{4\pi} S_\nu \, d\Omega - \int_{4\pi} I_\nu \, d\Omega \right\} d\nu$$

- so that

$$\frac{dF}{dz} = 4\pi \int_0^\infty \chi_\nu S_\nu \, d\nu - 4\pi \int_0^\infty \chi_\nu J_\nu \, d\nu$$

radiative equilibrium

- $\rightarrow$ in RE we have the additional condition
$$\int_0^\infty \chi_\nu S_\nu \, d\nu = \int_0^\infty \chi_\nu J_\nu \, d\nu$$

radiative equilibrium

- For $S_\nu = (1-\epsilon)J_\nu + \epsilon B_\nu$ this gives

$$\int_0^\infty \kappa_\nu J_\nu \, d\nu = \int_0^\infty \kappa_\nu B_\nu \, d\nu$$

- $\int_0^\infty \kappa_\nu J_\nu \, d\nu$ is the total energy absorbed by a volume element
- $\int_0^\infty \kappa_\nu B_\nu \, d\nu$ is the total energy emitted by a volume element

diffusion approximation

- consider $\tau \gg 1$ in a semi-infinite slab
- large depths
  - $\rightarrow$ photons trapped
  - $\rightarrow I_\nu$ nearly isotropic
  - $S_\nu \rightarrow B_\nu$
- use reference point $\tau_\nu \gg 1$
- expand S_ν around τ_ν

$$S(t_\nu) = \sum_{n=0}^{\infty} \frac{d^n B_\nu}{d\tau_\nu^n} \frac{(t_\nu - \tau_\nu)^n}{n!}$$

diffusion approximation

- insert series into FS for outgoing radiation ($\mu \geq 0$)

$$I_\nu(\tau_\nu, \mu) = \sum_{n=0}^{\infty} \mu^n \frac{d^n B_\nu}{d\tau_\nu^n} = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu} + \cdots$$

- similar for in-going radiation:
  - only differences are terms of $O(\exp(-\tau_\nu/\mu))$
  - $\rightarrow 0$ for $\tau_\nu \gg 1$
  - $\rightarrow$ above expansion is sufficient also for $\mu \leq 0$

diffusion approximation

- inserting this into equations for moments J, H, K $\rightarrow$

$$
\begin{aligned}
J_\nu(\tau_\nu) &= \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{d^{(2n)} B_\nu}{d\tau_\nu^{(2n)}} \\
&= B_\nu(\tau_\nu) + \frac{1}{3}\left(\frac{d^2 B_\nu}{d\tau_\nu^2}\right) + \cdots
\end{aligned}
$$

diffusion approximation

$$H_\nu(\tau_\nu) = \sum_{n=0}^{\infty} \frac{1}{2n+3} \frac{d^{(2n+1)} B_\nu}{d\tau_\nu^{(2n+1)}}$$

$$= \frac{1}{3} \left(\frac{dB_\nu}{d\tau_\nu} \right) + \cdots$$

$$K_\nu(\tau_\nu) = \sum_{n=0}^{\infty} \frac{1}{2n+3} \frac{d^{(2n)} B_\nu}{d\tau_\nu^{(2n)}}$$

$$= \frac{1}{3} B_\nu(\tau_\nu) + \frac{1}{5} \left(\frac{d^2 B_\nu}{d\tau_\nu^2} \right) + \cdots$$

diffusion approximation

- convergence of these series is very rapid
  - $d^n B_\nu / d\tau_\nu^n \approx B_\nu / \tau_\nu^n$
  - $\rightarrow$ ratio of successive terms is $O(1/\tau_\nu^2)$
- for large τ_ν we need only the leading terms

$$
\begin{aligned}
I_\nu(\tau_\nu) &= B_\nu(\tau_\nu) + \mu \left(\frac{dB_\nu}{d\tau_\nu} \right) \\
J_\nu(\tau_\nu) &= B_\nu(\tau_\nu) \\
H_\nu(\tau_\nu) &= \frac{1}{3} \left(\frac{dB_\nu}{d\tau_\nu} \right) \\
K_\nu(\tau_\nu) &= \frac{1}{3} B_\nu(\tau_\nu)
\end{aligned}
$$

diffusion approximation

$$H_\nu(\tau_\nu) \;=\; \frac{1}{3}\left(\frac{dB_\nu}{d\tau_\nu}\right)$$

- $\rightarrow$ diffusion equation for the radiative flux $F_\nu = 4H_\nu$

$$\begin{aligned} F_\nu &= \frac{4}{3}\left(\frac{dB_\nu}{d\tau_\nu}\right) \\ &= -\frac{4}{3}\left(\frac{1}{\chi_\nu}\frac{dB_\nu}{dT}\right)\left(\frac{dT}{dz}\right) \\ &= -D\nabla T \end{aligned}$$

diffusion approximation

- in addition

$$\frac{K_\nu(\tau_\nu)}{J_\nu(\tau_\nu)} = \frac{1}{3}$$

- $\rightarrow$ *Eddington approximation*