Stellar/Planetary Atmospheres

Part 01: description of radiation

Peter Hauschildt
yeti@hs.uni-hamburg.de

Hamburger Sternwarte
Gojenbergsweg 112
21029 Hamburg

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Topics

- The description of radiation
  - specific intensity
  - moments of the intensity
  - radiation field in TE
- additional literature:
  J. Oxenius, 'Kinetic Theory of Particles and Photons', Springer, 1986

The description of radiation

- Start with the *photon distribution function*

$$\Phi(\vec{r}, \vec{p}, t)$$

- average number of photons present at time t in $dV = (\vec{r}, d^3r)$ with momenta $d\vec{p} = (\vec{p}, d^3p)$
- neglected polarization
- completely analogous to description of particle gas
- differences:
  - photons have zero rest mass
  - no photon-photon collisions (0^{th} order)

specific intensity

- although Φ is perfectly good, historically the intensity is used
- transform $(\vec{p}, d^3p) \rightarrow (\nu, d\nu, \vec{n}, d\Omega)$
- use photon relations

$$\begin{aligned} E &= h\nu \\ p &= h\nu/v = E/c \end{aligned}$$

to *define* $I_\nu(\vec{n}, \vec{r}, t)$ through

$$\frac{dN_\nu}{dV} = \Phi(\vec{r}, \vec{p}, t)\, d^3p \equiv \frac{1}{h\nu c} I_\nu(\vec{n}, \vec{r}, t)\, d\nu d\Omega$$

where dN_ν/dV is the *photon density* ($[1/\text{cm}^3]$)

specific intensity

- with

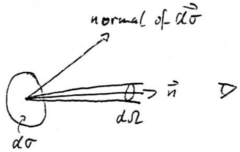
$$\vec{p} = \frac{h\nu}{c}\vec{n}$$
$$d^3p = p^2 dp d\Omega = \frac{h^3\nu^2}{c^3} d\nu d\Omega$$

we have

$$\frac{h^4}{c^2}\Phi(\vec{r},\vec{p},t) = \frac{I_\nu(\vec{n},\vec{r},t)}{\nu^3}$$

- $\rightarrow I_\nu/\nu^3$ is $\propto$ photon distribution function
- Remark: in a good approximation, $\Phi(\vec{r},\vec{p},t)$ is a *relativistic invariant*, i.e., I_ν/ν^3 is an invariant

meaning of the specific intensity



- consider

$$dE_\nu = I_\nu \cos\Theta \, d\sigma \, d\Omega \, dt \, d\nu$$

- radiation energy passing through surface element $d\vec{\sigma}$ in direction $\vec{n} = (\Theta, \varphi)$ in time interval dt and frequency interval $d\nu$ into solid angle interval $d\Omega$.
- I_ν is *independent* from the distance of an observer to the emitting surface
- unit of I_ν: $[\mathrm{erg/s/cm^2/sr/Hz}]$

meaning of the specific intensity

- I_ν defined per unit frequency interval $d\nu$
- transformation to unit wavelength interval $d\lambda$:
  - $\lambda = c/\nu \rightarrow d\lambda = -c/\nu^2\, d\nu$
  - therefore

$$I_\nu\, d\nu = -I_\lambda\, d\lambda$$

- unit of I_λ: $[\mathrm{erg/s/cm^2/sr/cm}]$ or $[\mathrm{erg/s/cm^2/sr/}]$

meaning of the specific intensity

- energy difference between unit $d\nu$ and $d\lambda$!
- $\rightarrow$ Shape of the same spectrum is *different* between I_λ and I_ν
- example: maximum of solar spectrum
  $\approx 4500\,\text{Å}$ for I_ν
  $\approx 8000\,\text{Å}$ for I_λ

I_λ vs. I_ν

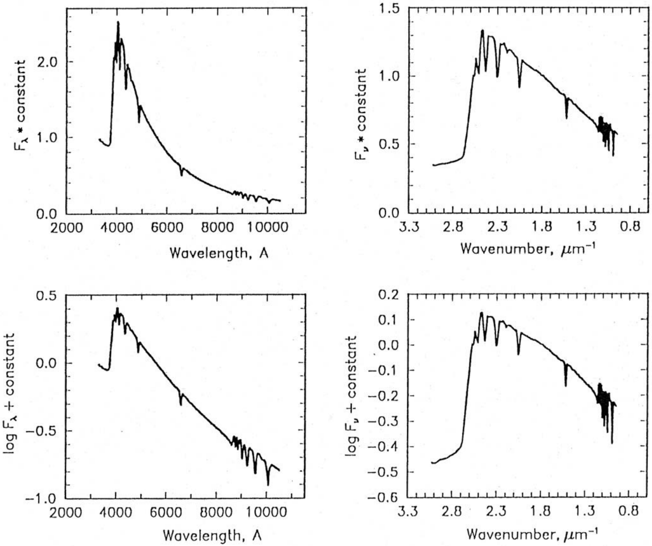


Fig. 10.4. The shape of a stellar energy distribution can look very different according to which coordinates are used. Those in Fig. 10.3 and the four shown here are the most common.

mean intensity J_ν

- average of I_ν with respect to $d\Omega$

$$J_\nu = \frac{1}{4\pi} \oint_{4\pi} I_\nu \, d\Omega$$

- $d\Omega = -d\cos\Theta \, d\varphi$
- if I_ν independent of φ (azimuthal angle!):
  - define $\mu = \cos\Theta \rightarrow$

$$J_\nu = \frac{1}{4\pi} \int \int I_\nu \, d\mu d\varphi = \frac{1}{2} \int_{-1}^{+1} I_\nu \, d\mu$$

mean intensity J_ν

- J_ν is $\propto$ radiation energy density E_ν

$$E_\nu = h\nu \oint_{4\pi} \frac{dN_\nu}{dV} \, d\Omega = \frac{1}{c} \oint_{4\pi} I_\nu \, d\Omega$$

therefore

$$E_\nu = \frac{4\pi}{c} J_\nu$$

- J_ν is the 0^{th} moment of I_ν

$$J_\nu = \frac{1}{2} \int \mu^0 I_\nu \, d\mu$$

radiation flux F_ν

- net flux of radiation through a surface element $d\sigma$:

$$\vec{F_\nu} = \oint_{4\pi} I_\nu \vec{n}\, d\Omega$$

- in 1D the vector $\vec{F_\nu}$ reduces to

$$F_\nu = \int \mu I_\nu \, d\mu d\varphi$$

so that (I_ν independent of φ!)

$$F_\nu = 2\pi \int \mu I_\nu \, d\mu$$

radiation flux F_ν

- first moment of I_ν with respect to μ:

$$H_\nu = \frac{1}{2}\int \mu I_\nu \, d\mu = \frac{1}{4\pi} F_\nu$$

- old literature: 'astrophysical flux'

$$\pi \mathcal{F} = F_\nu$$

Examples for F_ν

1. isotropic radiation (TE):

$$\vec{F_\nu} = \vec{0}$$

2. radiation flowing outside at top of plane parallel atmosphere:

$$F_\nu = 2\pi \int_0^1 \mu I_\nu \, d\mu$$

3. if, in addition, I_ν independent of μ:

$$F_\nu = \pi I_\nu$$

I_ν vs. F_ν

- I_ν independent of distance to radiating object
  $F_\nu \propto 1/r^2$
- I_ν can be measured only for *resolved* sources (solid angle!)
- unresolved sources (most stars) $\rightarrow$
  only F_ν can be measured

radiation pressure

$$P_\nu = \frac{1}{c} \oint \vec{n} I_\nu \vec{n} \, d\Omega$$

- P_ν: radiation pressure tensor
- diagonal elements: radiation pressure *normal* to x, y, or z surface
- off-diagonal elements $\rightarrow$ *shear forces*
- 1D case:

$$P_\nu = \frac{1}{c} \int \mu^2 I_\nu \, d\mu d\varphi = \frac{2\pi}{c} \int_{-1}^{+1} \mu^2 I_\nu \, d\mu$$

radiation pressure

- second moment of I_ν:

$$K_\nu = \frac{1}{2}\int \mu^2 I_\nu \, d\mu$$

so that

$$P_\nu = \frac{4\pi}{c} K_\nu$$

radiation field in TE

- derive TE value of radiation field $\rightarrow$ use photon picture and *detailed balance*
  1. stimulated and spontaneous processes must be considered
  2. only consider absorptive processes (change photon number density!)
- consider 'reaction' (bremsstrahlung):

$$Q + P(E, dE) \rightleftharpoons Q + P(E - h\nu, dE) + \gamma(\nu, d\nu)$$

  - Q: heavy (resting) particle
  - P: radiating particle (e.g., electron)

radiation field in TE

- define $N(\nu)$: number of photons in $(\nu, d\nu)$ integrated over all directions
- define $G(\nu)$: number of possible photon stages in $(\nu, d\nu)$
- define
$$z(\nu) = \frac{N(\nu)d\nu}{G(\nu)d\nu} = \frac{N(\nu)}{G(\nu)}$$
- analogous definition for P!

radiation field in TE

- reaction rate of
$$A + B \to C + D$$
is proportional to
  1. $N(A) \times N(B)$ (how many chances?)
  2. $G(C) \times G(D)$ (how many reaction channels)
- Bosons B: $1 + \frac{N(B)}{G(B)}$
- Fermions F: $1 - \frac{N(F)}{G(F)}$

radiation field in TE

- detailed balance $\rightarrow$

$$N(E)G(E-h\nu)G(\nu)[1+z(\nu)] = N(E-h\nu)N(\nu)G(E)$$

so that

$$\frac{1+z(\nu)}{z(\nu)} = \frac{z(E-h\nu)}{z(E)} = \exp\left(\frac{h\nu}{kT}\right)$$

(Boltzmann!)

radiation field in TE

- therefore

$$z(\nu) = \frac{1}{\exp(h\nu/kT) - 1}$$

- or

$$N(\nu)d\nu = G(\nu)\frac{1}{\exp(h\nu/kT) - 1}\,d\nu$$

- $G(\nu)$ is given by

$$G(\nu)d\nu = 2V\frac{4\pi p^2 dp}{h^3} = V\frac{8\pi\nu^2}{c^3}\,d\nu$$

- factor 2 $\rightarrow$ polarization
- used $p = h\nu/c$

radiation field in TE

- used photon density $n(\nu) = N(\nu)/V$ gives

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{1}{\exp(h\nu/kT) - 1}\,d\nu$$

  is the TE value of the photon density (Planck)
- with $E_\nu = 4\pi/cJ_\nu$

$$J_\nu = I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}$$

  Planck function
  black body radiation

Planck function

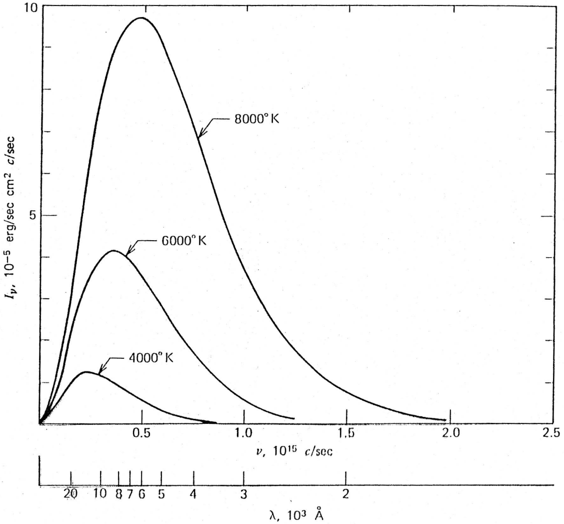


Fig. 6.2. The specific intensity emitted by a black body is shown for the three temperatures indicated. The scaling of Wein's law (eq. (6.1)) can be seen in the shape of the curves. The area under the curves obeys the Stefan–Boltzmann law (eq. (6.3)).

B_ν numerics

- wavelength scale:

$$B_\lambda(T,\lambda)\,d\lambda = \frac{2hc^2}{\lambda^5}\frac{d\lambda}{\exp(hc/\lambda kT)-1} = \frac{c_1}{\lambda^5}\frac{d\lambda}{\exp(c_2/\lambda T)-1}$$

- ν scale:

$$B_\nu(T,\nu)\,d\nu = c_3\nu^3\frac{1}{\exp(c_4\nu/T)-1}\,d\nu$$

B_ν numerics

- for λ in [Å] and B_λ in $[\mathrm{erg/s/cm^2/sr/}]$:
  - $c_1 = 2hc^2 = 1.19 \times 10^{27}$
  - $c_2 = hc/k = 1.44 \times 10^8$
- for ν in [Hz] and B_ν in $[\mathrm{erg/s/cm^2/sr/Hz}]$
  - $c_3 = 2h/c^2 = 1.47 \times 10^{-47}$
  - $c_4 = h/c = 4.8 \times 10^{-11}$

B_ν approximations

- $h\nu/kT \ll 1$ (Rayleigh-Jeans):

$$B_\nu \rightarrow \frac{2kT\nu^2}{c^2}$$

- $h\nu/kT \gg 1$ (Wien):

$$B_\nu \rightarrow \frac{2h\nu^3}{c^2} \exp(-h\nu/kT)$$

Wien's law

- maximum of B_λ:

$$\lambda_m T = \text{const.} \approx 5.1 \times 10^7\,\text{Å K}$$

- maximum of B_ν:

$$\lambda_m T = \text{const.} \approx 2.9 \times 10^7\,\text{Å K}$$

Stefan-Boltzmann law

- flux emerging from a very small 'hole' in a black body is given by
$$F_\nu = \pi I_\nu = \pi B_\nu$$
- total flux $\rightarrow$

$$F(T) = \pi \int_0^\infty B_\nu(T,\nu)\, d\nu = \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{\exp(x) - 1}\, dx$$

with $x = h\nu/kT$

Stefan-Boltzmann law

- $\int_0^\infty \frac{x^3}{\exp(x)-1}\,dx = \pi^4/15$
- so that
$$F(T) = \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 = \sigma T^4$$
- $\sigma = 5.67 \times 10^{-5}\,\mathrm{erg/cm^2/s/K^4}$: Stefan's constant